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ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS AND REACTION-DIFFUSION SYSTEMS

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ABSTRACT. Several fundamental results on the existence and behavior of solutions to semilinear functional differential equations are developed in a Banach space setting. The ideas are applied to reaction-diffusion systems that have time delays in the nonlinear reaction terms. The techniques presented here include differential inequalities, invariant sets, and Lyapunov functions, and therefore they provide for a wide range of applicability. The results on inequalities and especially strict inequalities are new even in the context of semilinear equations whose nonlinear terms do not contain delays.

Suppose Ω is a bounded domain in $\mathbf{R}^N$ with $\partial\Omega$ smooth and Δ is the Laplacian operator on Ω. Also, let m be a positive integer, τ a positive number, and $f = (f_i)_1^m$ a continuous, bounded function from $[0, \infty] \times \overline{\Omega} \times C([-\tau, 0])^m$ into $\mathbf{R}^m$ where $C([-\tau, 0])$ is the space of continuous functions from $[-\tau, 0]$ into $\mathbf{R}$. The purpose of this paper is to apply abstract results for semilinear functional differential equations in Banach spaces to reaction-diffusion systems with time delays having the form

$$\text{(RDD)}\quad \begin{aligned} &\partial_t u^i(x, t) = d_i \Delta u^i(x, t) + f_i(t, x, u_t(x, \cdot)), \\ &\qquad\qquad t > a, \ x \in \Omega, \ i = 1, \dots, m, \\ &\alpha_i(x) u^i(x, t) + \partial_n u^i(x, t) = \beta_i(x, t), \\ &\qquad\qquad t > a, \ x \in \partial\Omega, \ i = 1, \dots, m, \\ &u^i(x, a + \theta) = \chi^i(x, \theta), \quad -\tau \le \theta \le 0, \ x \in \Omega, \ i = 1, \dots, m, \end{aligned}$$

where $a \geq 0$, $d_i \geq 0$, and $\alpha_i : \overline{\Omega} \to [0, \infty)$ is C^1 and $\beta_i : \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ is C^2. Here ∂_n is the outward normal derivative on $\partial\Omega$ and if $d_i = 0$ it is assumed that no boundary conditions are specified for this i. Also, $\partial_t u^i(x, t)$ denotes the partial with respect to t, whereas $u_t(x, \cdot)$ denotes the member of $C([-\tau, 0])$ defined by $\theta \to u(x, t + \theta) = (u^i(x, t + \theta))_1^m$.

Our techniques provide basic existence criteria, but the main point is that they can also be effectively applied to obtain estimates for solutions, especially

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those pertaining to inequalities and asymptotic behavior. In fact, the abstract theorems may be used to analyze solutions to (RDD) with differential inequalities, invariant sets, and Lyapunov-like functions. Therefore, these methods are quite flexible and have a wide range of applicability.

The abstract results deal with semilinear integral equations of the form

$$\text{(SIE)} \qquad \begin{aligned} &u(t) = S(t, a)\chi(0) + \int_a^t T(t, r)B(r, u_r)\,dr, \qquad t \geq a, \\ &u(a+\theta) = \chi(\theta), \qquad -\tau \leq \theta \leq 0. \end{aligned}$$

Here $T = \{T(t, s): t \geq s \geq a\}$ is a linear evolution system on a Banach space X and $S = \{S(t, s): t \geq s \geq a\}$ is an affine evolution system that is a nonhomogeneous perturbation of T. The function B is continuous from $[a, \infty) \times \mathscr{C}$ into X where $\mathscr{C}$ is the space of continuous functions from $[-\tau, 0]$ into X. In (SIE) $\chi \in \mathscr{C}$ is the given initial function and for each $r \in [a, t]$, u_r denotes the member of $\mathscr{C}$ defined by $u_r(\theta) = u(r+\theta)$ for $\theta \in [-\tau, 0]$. For the past decade there has been much research on abstract integral equations included in the form of (SIE). Basic results are contained in Fitzgibbon [1], Lightbourne [8], Rankin [15], Travis and Webb [19], and Webb [21]. Techniques for functional differential equations in Banach spaces [i.e., when $S(t, s)x \equiv T(t, s)x \equiv x$ for all $x \in X$ and $t \geq s \geq a$] can be found in Leela and Moauro [6], Lightbourne [7], and Seifert [16]. Ideas related to the methods presented here but pertaining to semilinear systems whose nonlinear term contains no delays are developed in Martin [10] and in the books by Lakshmikantham and Leela [5], Martin [9], and Smoller [18].

One of the main techniques presented here is the development of abstract differential and integral inequalities. Very general and effective results are obtained for (SIE) under the assumption that the ordering makes X into a Banach lattice. Extensions of the concept of strict inequalities are also given in this general setting and are based on estimates involving positive linear functionals. Previous results for order-preserving properties for functional differential equations were developed in Kunisch and Schappacher [4], Martin [13], and Smith [17]. The proof techniques for strict inequalities are motivated by the ideas in Martin [11–13] and Smith [17].

The organization of this paper is as follows: The principal results for (RDD) involving existence, invariance, and inequalities for solutions are described in §1. These ideas form the core of applicability for our abstract techniques. §2 introduces notation and states the fundamental abstract results on existence and invariance for (SIE) (the proofs are given in the final section). Integral inequalities for (SIE) in a Banach lattice setting are also developed in §2 and the continuation of these ideas to strict inequalities and systems is given in §3. In §4 we give a detailed proof of the main existence and invariance results stated in §2.

1. Reaction-diffusion systems with delay

Suppose Ω is a bounded region in $\mathbf{R}^N$ with $\partial\Omega$ smooth, Δ is the Laplacian operator on Ω, and ∂_n is the outward normal derivative on $\partial\Omega$. Suppose also that $C([-\tau, 0])$ is the space of continuous functions from $[-\tau, 0]$ into $\mathbf{R}$ with the supremum noun and that $f = (f_i)_1^m$ is a continuous function from $[0, \infty) \times \overline{\Omega} \times C([-\tau, 0])^m$ into $\mathbf{R}^m$ which is bounded on bounded sets. In this section we state our main results regarding the behavior of solutions to a reaction-diffusion system with delays having the form

$$(1.1)\qquad \begin{aligned} &\partial_t u^i(x, t) = d_i \Delta u^i(x, t) + f_i(t, x, u_t^1(x, \cdot), \dots, u_t^m(x, \cdot)), \\ &\qquad\qquad\qquad\qquad t > a, \; x \in \Omega, \\ &\alpha_i(x) u^i(x, t) + k_i \partial_n u^i(x, t) = \beta_i(x, t), \qquad t > a, \; x \in \partial\Omega, \\ &u^i(x, a + \theta) = \chi^i(x, \theta), \qquad -\tau \le \theta \le 0, \; x \in \Omega, \end{aligned}$$

where $a \ge 0$ and $i = 1, \dots, m$. It is assumed that the coefficients in (1.1) satisfy the following:

(1.2)

(a) There is a subset Σ_0 of $\{1, \dots, m\}$ such that $d_i = 0$ for all $i \in \Sigma_0$ and $d_i > 0$ for all $i \in \Sigma_0^c$.

(b) $\alpha_i : \overline{\Omega} \to [0, \infty)$ is C^1 and $\beta_i : \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ is C^2 for $i = 1, \dots, m$.

(c) If $i \in \Sigma_0^c$ then $k_i = 1$ and if $i \in \Sigma_0$ then $\alpha_i = 0$, $d_i \equiv 0$, and $\beta_i \equiv 0$

Observe that if $i \in \Sigma_0$ then the ith equation in (1.1) is an ordinary functional differential equation with the ith component depending on the "parameter" $x \in \overline{\Omega}$. We allow the extreme cases $\Sigma_0 = \varnothing$ and $\Sigma_0 = \{1, \dots, m\}$. Under the assumptions in (1.2) it is often convenient to write (1.1) into two separate parts:

(1.3)

$$\begin{aligned} &\partial_t u^i(x, t) = f_i(t, x, u_t^1(x, \cdot), \dots, u_t^m(x, \cdot)), \qquad t > a, \; x \in \Omega, \; i \in \Sigma_0, \\ &u^i(x, a + \theta) = \chi^i(x, \theta), \qquad -\tau \le \theta \le 0, \; x \in \overline{\Omega}, \; i \in \Sigma_0, \end{aligned}$$

and

$$(1.4)\qquad \begin{aligned} &\partial_t u^i(x, t) = d_i \Delta u^i(x, t) + f_i(t, x, u_t^1(x, \cdot), \dots, u_t^m(x, \cdot)), \\ &\qquad\qquad\qquad\qquad t > a, \; x \in \Omega, \; i \in \Sigma_0^c, \\ &\alpha_i(x) u^i(x, t) + \partial_n u^i(x, t) = \beta_i(x, t), \qquad t > a, \; x \in \partial\Omega, \; i \in \Sigma_0^c, \\ &u^i(x, a + \theta) = \chi^i(x, \theta), \qquad -\tau \le \theta \le 0, \; x \in \overline{\Omega}, \; i \in \Sigma_0^c. \end{aligned}$$

The initial values χ^i are assumed to be continuous on $\overline{\Omega} \times [-\tau, 0]$.

We now give the basic assumptions on the nonlinear term $f = (f_i)_1^m$. It is assumed that Λ is a closed convex subset of $\mathbf{R}^m$ and for each $\xi \in \mathbf{R}^m$ define

$$d(\xi; \Lambda) = \inf\{|\xi - \eta| : \eta \in \Lambda\}$$

where $|\cdot|$ is some norm on $\mathbf{R}^m$. Also, set

$$C_\Lambda^m = \{\varphi \in C([-\tau, 0])^m : \varphi(\theta) \in \Lambda \text{ for all } -\tau \le \theta \le 0\}$$

and note that C_Λ^m is a closed convex set. The underlying assumptions on f are as follows:

(a) f is continuous from $[0, \infty) \times \overline{\Omega} \times C_\Lambda^m$ into $\mathbf{R}^m$.

(b) For each $R > 0$ there exist $\nu = \nu(R) \in (0, 1]$ and $L = L(R) \in (0, \infty)$ such that

$$|f_i(t, x, \varphi) - f_i(s, x, \psi)| \le L\left(|t-s|^\nu + \sum_{j=1}^m |\varphi_j - \psi_j|\right) \tag{1.5}$$

for all $t, s \in [0, R]$, $x \in \overline{\Omega}$, $\varphi, \psi \in C([-\tau, 0])^m$ with $\|\varphi\|, \|\psi\| \le R$, and $i = 1, \ldots, m$.

(c)

$$\lim_{h\to 0+} \frac{1}{h} d(\varphi(0) + hf(t, x, \varphi); \Lambda) = 0 \quad \text{for all } (t, x, \varphi) \in [0, \infty) \times \overline{\Omega} \times C_\Lambda^m.$$

Assumption (1.5c) is a subtangential condition on f relative to the set Λ and has been employed in the study of functional differential equations by several authors (see, for example, Seifert [16], Leela and Moauro [6], and Lightbourne [7]).

Even when the function f is smooth, if Σ_0 is a proper subset of $\{1, \ldots, m\}$, then system (1.1) may not have a solution in a classical sense. Therefore, it is necessary to consider generalized solutions to (1.1) and so we use ideas from the theory of C_0 semigroups of bounded linear operators in a Banach space (see Goldstein [3] or Pazy [14]). In particular, consider the uncoupled linear system

$$\begin{aligned} &\partial_t v^i(x, t) = d_i \Delta v^i(x, t), \qquad t > 0,\ x \in \Omega,\ i \in \Sigma_0^c, \\ &\alpha_i(x) v^i(x, t) + \partial_n v^i(x, t) = 0, \qquad t > 0,\ x \in \partial\Omega,\ i \in \Sigma_0^c, \\ &v^i(x, 0) = v_0^i(x), \qquad x \in \overline{\Omega},\ i \in \Sigma_0^c, \end{aligned} \tag{1.6}$$

where $v_0^i \in C(\overline{\Omega})$ and $C(\overline{\Omega})$ is the space of continuous real-valued functions on $\overline{\Omega}$ with the supremum norm $|\cdot|_\infty$. Let $C(\overline{\Omega})^m$ be the product Banach space of continuous functions $y = (y_i)_1^m$ from $\overline{\Omega}$ into $\mathbf{R}^m$ with

$$|y|_\infty \equiv \max\{|y_i|_\infty : i = 1, \ldots, m\}$$

and for each $t > 0$ define the family of linear operators $T(t) = (T_i(t))_1^m$ on $C(\overline{\Omega})^m$ in the following manner:

(1.7) for each $v_0 = (v_0^i)_1^m \in C(\overline{\Omega})^m$ define $T(t)v_0 = (T_i(t)v_0^i)_1^m$ where $T_i(t)v_0^i = v_0^i$ if $i \in \Sigma_0$ and $T_i(t)v_0^i = v^i(\cdot, t)$ with v^i the solution to (1.6) if $i \in \Sigma_0^c$.

It is well known that T is a C_0 semigroup on $C(\overline{\Omega})^m$ that is nonexpansive and analytic. In order to include the nonhomogeneous boundary conditions in (1.1), we also consider the system

$$(1.8)\quad \begin{aligned} &\partial_t v^i(x,t) = d_i \Delta v^i(x,t), \qquad t > s \geq 0,\ x \in \Omega,\ i \in \Sigma_0^c, \\ &\alpha_i(x) v^i(x,t) + \partial_n v^i(x,t) = \beta_i(x,t), \qquad t > s \geq 0,\ x \in \partial\Omega,\ i \in \Sigma_0^c, \\ &v^i(x,s) = v_0^i(x), \qquad s \geq 0,\ x \in \overline{\Omega},\ i \in \Sigma_0^c, \end{aligned}$$

and define the family $S(t,s) = (S_i(t,s))_1^m$, $t \geq s \geq 0$, of affine operators on $C(\overline{\Omega})^m$ by

$$(1.9)\quad \begin{aligned} &S(t,s)v_0 = (S_i(t,s)v_0^i)_1^m \text{ where } S_i(t,s)v_0^i = v_0^i \text{ if } i \in \Sigma_0 \text{ and} \\ &S_i(t,s)v_0^i = v^i(\cdot,t) \text{ with } v^i \text{ the solution to (1.8) if } i \in \Sigma_0^c. \end{aligned}$$

Using standard arguments, it follows that if $i \in \Sigma_0^c$ and $\gamma^i : \overline{\Omega} \times [0,\infty) \to \mathbf{R}$ is smooth with

$$\alpha_i(x)\gamma^i(x,t) + \partial_n \gamma^i(x,t) = \beta_i(x,t) \quad \text{on } \partial\Omega \times (0,\infty),$$

then S and T are connected by the formula

$$S_i(t,s)v_0^i = T_i(t-s)[v_0^i - \hat{\mu}_i(s)] + \hat{\mu}_i(t), \qquad t \geq s \geq 0,\ v_0^i \in C(\overline{\Omega}),$$

where $\hat{\mu}_i \equiv 0$ if $i \in \Sigma_0$ and

$$\hat{\mu}_i(\cdot,t) = \gamma^i(\cdot,t) + \int_0^t T_i(t,r)[d_i \Delta \gamma^i(\cdot,r) - \partial_t \gamma^i(\cdot,r)]\,dr$$

if $i \in \Sigma_0^c$ [see property (S3) in §2].

Finally, let $\mathscr{C} \equiv \mathscr{C}([-\tau,0]; C(\overline{\Omega})^m)$ be the space of continuous functions from $[-\tau,0]$ into $C(\overline{\Omega})^m$ and identify members φ of $\mathscr{C}$ as functions from $\overline{\Omega} \times [-\tau,0]$ into $\mathbf{R}^m$: $\varphi(x,\theta) \equiv [\varphi(\theta)](x)$. In particular, φ is continuous on $\overline{\Omega} \times [-\tau,0]$ and it follows that $\varphi(x,\cdot) \in C([-\tau,0];\mathbf{R}^m)$ for each $x \in \overline{\Omega}$. Thus $f(t,x,\varphi(x,\cdot))$ is well defined on $[0,\infty) \times \overline{\Omega} \times \mathscr{C}$ and, by continuity, $x \to f(t,x,\varphi(x,\cdot))$ is in $C(\overline{\Omega})^m$ whenever $\varphi(x,\theta) \in \Lambda$ for all $(x,\theta) \in \overline{\Omega} \times [-\tau,0]$ [see (1.5a)]. Therefore, if

$$(1.10)\quad \begin{aligned} &\mathscr{C}_\Lambda \equiv \{\varphi \in \mathscr{C} : \varphi(x,\theta) \in \Lambda \text{ for all } (x,\theta) \in \overline{\Omega} \times [-\tau,0]\} \\ &\text{and } [B_i(t,\varphi)](x) \equiv f_i(t,x,\varphi(x,\cdot)) \text{ for } (t,\varphi) \in [0,\infty) \times \mathscr{C}_\Lambda, \\ &x \in \overline{\Omega}, \text{ and } i = 1,\ldots,m, \end{aligned}$$

then B is a continuous function from $[0,\infty) \times \mathscr{C}_\Lambda$ into $C(\overline{\Omega})^m$.

Let $T = (T_i)_1^m$ be as in (1.7), $S = (S_i)_1^m$ as in (1.9), $B = (B_i)_1^m$ as in (1.10), and consider the system of integral equations having the form

$$(1.11)\quad u^i(t) = S_i(t,a)\chi_i(a) + \int_a^t T_i(t-r)B_i(r,u_r)\,dr, \qquad u_a^i = \chi_i,\ t \geq a,$$

where $i = 1,\ldots,m$ and u_r denotes the member φ of $\mathscr{C}(\overline{\Omega})^m$ defined by $\varphi(\theta) = (u^i(r+\theta))_1^m$ for $-\tau \leq \theta \leq 0$. Under these circumstances, if $(u^i)_1^m$ is a smooth solution to (1.1) on $\overline{\Omega} \times [a, b]$ and $[u^i(t)](x) \equiv u^i(x, t)$ for $(x, t) \in \overline{\Omega} \times [a, b]$ and $i = 1, \ldots, m$, then $u = (u^i)_1^m$ is a solution to (1.11) for $t \in [a, b]$. Conversely, if $(u^i)_1^m$ is a solution to (1.11) on $[a, b]$ having the property that the functions $u^i(x, t) \equiv [u^i(t)](x)$ are C^1 in t and C^2 in x for $i \in \Sigma_0^c$, then $(u^i)_1^m$ is also a solution to (1.1). Therefore, a solution $(u^i)_1^m$ to the abstract integral equation (1.11) is said to be a *mild solution* to (1.1). Here it is understood that mild solutions are considered in the space $C(\overline{\Omega})^m$—that is, S, T, B, and the integral in (1.11) are defined in terms of the space $C(\overline{\Omega})^m$ [see equation (2.1) with $X = C(\overline{\Omega})^m$].

The subtangential condition (1.5c) on f implies that for each $x \in \overline{\Omega}$, the solution w to

$$w'(t) = f(t, x, w_t), \qquad w_a = w^0, \quad t \geq a,$$

satisfies $w(t) \in \Lambda$ for $t \geq a$ whenever $w^0(\theta) \in \Lambda$ for all $-\tau \leq \theta \leq 0$ (see, e.g., Seifert [16]). Therefore, our final assumption asserts that a similar property for the linear part of (1.1) is valid:

(1.12)

$$v_0(x) \in \Lambda \text{ for all } x \in \overline{\Omega} \text{ implies } [S(t, s)v^0](x) \in \Lambda \text{ for all } t \geq s \geq 0 \text{ and } x \in \overline{\Omega}.$$

Stated in other terms, (1.12) implies that if $v_0(x) \in \Lambda$ for all $x \in \overline{\Omega}$, $(v^i)_1^m$ is defined on $\overline{\Omega} \times [a, \infty)$ by $v^i(x, t) \equiv v_0^i(x)$ for $i \in \Sigma_0$, and v^i is the solution to (1.8) for $i \in \Sigma_0^c$, then $(v^i(x, t))_1^m \in \Lambda$ for all $(x, t) \in \overline{\Omega} \times [s, \infty)$.

Under these conditions we have the following basic result on the existence and uniqueness of a solution to (1.1):

Theorem 1. *Suppose that* (1.2), (1.5), *and* (1.12) *are satisfied. Then* (1.1) *has a unique noncontinuable mild solution* $u = (u^i)_1^m$ *defined on* $\overline{\Omega} \times [a - \tau, b)$ *where* $b = b(\chi)$ *and* $a < b \leq \infty$. *Furthermore,* $u(x, t) \in \Lambda$ *for all* $(x, t) \in \overline{\Omega} \times [a, b)$ *and if* $b < \infty$ *then* $\|u_t\|_\infty \to \infty$ *as* $t \to b^-$. *Also, if* $b > a + \tau$ *then* $(u^i)_1^m$ *is a classical solution to* (1.1) *for* $(x, t) \in \overline{\Omega} \times [a + \tau, b)$.

The existence of a unique noncontinuable mild solution u to (1.1) having values in Λ is a direct consequence of our existence and invariance theorem for abstract integral equations of the form (1.11), which is given in the next section (see Theorem 2). In order to establish the differentiability of the mild solution u, note that each $S_i(t, a)\chi_i(0)$ is C^1 for $t > a$ since the boundary terms are smooth. Thus if A is the (infinitesimal) generator of T and $g(t) \equiv B(t, u_t)$ for $a \leq t < b$, then g is continuous and $u(t) = S(t, a)\chi(0) + v(t)$ where

$$v(t) = \int_a^t T(t - r)g(r)\, dr \quad \text{for } a \leq t < b.$$

In particular, v is the mild solution $v' = Av + g(t)$, $v(a) = 0$, and hence v is Hoelder continuous on (a, b) by Theorem 3.1 of Pazy [14, p. 110]. The Lipschitz continuity of f is then seen to imply that $g(t)$ is Hoelder continuous on

$(a+\tau, b)$ (since u_t depends only on the values of u on $[t-\tau, t]$). Therefore, by Theorem 3.2 of Pazy [14, p. 311], v is C^1 on $(a+\tau, b)$, $v(t) \in D(A)$, and

$$v'(t) = Av(t) + g(t) \quad \text{for } a+\tau < t < b.$$

From this and the smoothness of S it follows that u is C^1 on $(a+\tau, b)$, $u(t) - \hat{\mu}(t) \in D(A)$, and

$$u'(t) = A(u(t) - \hat{\mu}(t)) + \hat{\mu}(t) + B(t, u_t).$$

But $[u(t)](x) = u(x, t)$, so $u'(t)$ has the representation

$$[u'(t)](x) = \lim_{h\to 0} \frac{u(x, t+h) - u(x, t)}{h} = \partial_t u(x, t)$$

and the limit is uniform for $x \in \overline{\Omega}$. From these observations one can now deduce that u is a classical solution to (1.1) on $(a+\tau, b)$.

Theorem 1 is applicable in many important situations, and we list several here in the form of remarks.

Remark 1.1. If $\Lambda = [0, \infty)^m$ then (1.12) holds only in case $\beta_i \geq 0$ on $\partial\Omega \times [0, \infty)$ for all $i \in \Sigma_0^c$ by the maximum principle. Also, (1.5c) holds only in case $f = (f_i)_1^m$ is quasipositive: if $k \in \{1, \dots, m\}$ and $(t, \varphi) \in [0, \infty) \times C([-\tau, 0])^m$ with $\varphi_i(\theta) \geq 0$ for all $-\tau \leq \theta \leq 0$ and $i = 1, \dots, m$, then $\varphi_k(0) = 0$ implies $f_k(t, x, \varphi) \geq 0$ for all $x \in \overline{\Omega}$. Thus Theorem 1 gives criteria to determine if solutions to (1.1) remain nonnegative if they are nonnegative initially.

Remark 1.2. Theorem 1 also includes the method of invariant rectangles for (1.1). For suppose that

$$\Lambda = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m]$$

where $-\infty \leq a_i < b_i \leq +\infty$. Then the maximum principle implies that (1.12) holds whenever

$$\alpha_i(x)a_i \leq \beta_i(x, t) \leq \alpha_i(x)b_i \quad \text{for } (x, t) \in \partial\Omega \times [0, \infty), \ i \in \Sigma_0^c,$$

where $\alpha_i(x)(-\infty) \equiv -\infty$ and $\alpha_i(x)(+\infty) \equiv +\infty$. Furthermore, it is easy to check directly that (1.5c) holds only in case $f = (f_i)_1^m$ has the following property:

> If $k \in \{1, \dots, m\}$ and $\varphi = (\varphi_i)_1^m \in C([-\tau, 0])^m$ with $a_i \leq \varphi_i(\theta) \leq b_i$ for all $-\tau \leq \theta \leq 0$ and $i = 1, \dots, m$, then $\varphi_k(0) = a_k$ implies $f_k(t, x, \varphi) \geq 0$ for all $(t, x) \in [0, \infty) \times \overline{\Omega}$ and $\varphi_k(0) = b_k$ implies $f_k(t, x, \varphi) \leq 0$ for all $(t, x) \in [0, \infty) \times \overline{\Omega}$.

Remark 1.3. If the diffusion coefficients are equal in (1.1) then Theorem 1 applies to a wide variety of convex sets Λ. So assume that Γ is an indexing set and that the following hold:

(a) $\{\eta_\sigma : \sigma \in \Gamma\} \subset \mathbf{R}^m$ and $\{\gamma_\sigma : \sigma \in \Gamma\} \subset \mathbf{R}$ are such that $\xi \in \Lambda \Leftrightarrow \eta_\sigma \cdot \xi \leq \gamma_\sigma$ for all $\sigma \in \Gamma$.

(b) $d_i \equiv d$ and $\alpha_i \equiv \alpha$ for all $i = 1, \dots, m$.

Then (1.12) is valid whenever

$$\eta_\sigma \cdot (\beta_i(x, t))_1^m \le \alpha(x)\gamma_\sigma \quad \text{for all } \sigma \in \Gamma \text{ and } (x, t) \in \partial\Omega \times [0, \infty)$$

(notice that this is always satisfied if $\beta_i \equiv \alpha_i \equiv 0$—that is, each of the boundary conditions is homogeneous and Neumann). For suppose $y = (y_i)_1^m \in C(\overline{\Omega})^m$ with $y(x) \in \Lambda$ for all $x \in \overline{\Omega}$ and define $\delta(x, t) = \eta_\sigma \cdot [S(t, s)y](x)$ on $\overline{\Omega} \times [s, \infty)$. Then

$$\begin{aligned} &\partial_t \delta(x, t) = d\Delta\delta(x, t) \quad [\text{since } d_i \equiv d], \\ &\alpha(x)\delta(x, t) + \partial_n \delta(x, t) \le \alpha(x)\gamma_\sigma \quad [\text{since } \alpha_i \equiv \alpha), \\ &\delta(x, s) = \eta_\sigma \cdot y(x) \le \gamma_\sigma \quad [\text{since } y(x) \in \Lambda]. \end{aligned}$$

The maximum principle implies that $\delta(x, t) \le \gamma_\sigma$, and so (1.12) must hold. Condition (1.15c) is valid in this case if and only if

> whenever $(t, x, \varphi) \in [0, \infty) \times \overline{\Omega} \times C([-\tau, 0])^m$ and $\sigma \in \Gamma$ is such that $\eta_\sigma \cdot \varphi(0) = \gamma_\sigma$, then $\eta_\sigma \cdot f(t, x, \varphi) \le 0$.

(A proof of this is in [9, Lemma 7.3, p. 258], for example.) Of course, if all the d_i's are zero, this is the standard invariance result for functional differential equations [16].

Our final results are based on inequalities for solutions. If $u = (u^i)_1^m$ and $v = (v^i)_1^m$ are $\mathbf{R}^m$-valued functions in $\overline{\Omega} \times I$, we write $u \le v$ whenever $u^i(x, t) \le v^i(x, t)$ for all $(x, t) \in \overline{\Omega} \times I$ and $i = 1, \ldots, m$. Also, if $\xi = (\xi_i)_1^m$ and $\eta = (\eta_i)_1^m$ are in $\mathbf{R}^m$, we write $\xi \le \eta$ whenever $\xi_i \le \eta_i$ for all $i = 1, \ldots, m$. Moreover, if $\xi^+, \xi^- \in \mathbf{R}^m$ with $\xi^- \le \xi^+$, then $[\xi^-, \xi^+] \equiv \{\eta \in \mathbf{R}^m : \xi^- \le \eta \le \xi^+\}$ (that is, $[\xi^-, \xi^+] \equiv \prod_{i=1}^m [\xi_i^-, \xi_i^+]$).

Under appropriate circumstances, we show that the existence of an upper and a lower solution to (1.1) implies the existence of a solution to (1.1) lying in between the upper and lower solutions. So suppose $v^\pm = (v_i^\pm)_1^m$ are continuously differentiable functions from $\overline{\Omega} \times [a - \tau, c)$ into Λ where $a < c \le \infty$, that they are C^2 in $x \in \Omega$, $i \in \Sigma_0^c$, and that

$$v^-(x, t) \le v^+(x, t) \quad \text{and} \quad [v^-(x, t), v^+(x, t)] \subset \Lambda \quad \text{for } (x, t) \in \overline{\Omega} \times [a - \tau, c).$$

Furthermore, let $f^\pm = (f_i^\pm)_1^m$ be continuous functions from $[0, \infty) \times \overline{\Omega} \times C([0, \tau])^m$ into $\mathbf{R}^m$ and assume the following differential inequalities are satisfied:

$(1.15)_+$

$$\begin{aligned} &\partial_t v_i^+(x, t) \ge d_i \Delta v_i^+(x, t) + f_i^+(t, x, v_t^+(x, \cdot)), \qquad a < t < c, \ x \in \Omega, \\ &\alpha_i(x) v_i^+(x, t) + \partial_n v_i^+(x, t) = \beta_i^+(x, t) \ge \beta_i(x, t), \qquad a < t < c, \ x \in \partial\Omega, \\ &v_i^+(x, a + \theta) = \chi_i^+(x, \theta) \ge \chi^i(x, \theta), \qquad -\tau \le \theta \le 0, \ x \in \Omega, \end{aligned}$$

and

$(1.15)_-$

$$\begin{aligned}
&\partial_t v_i^-(x, t) \le d_i \Delta v_i^-(x, t) + f_i^-(t, x, v_t^-(x, \cdot)), \qquad a < t < c, \ x \in \Omega, \\
&\alpha_i(x) v_i^-(x, t) + \partial_n v_i^-(x, t) = \beta_i^-(x, t) \le \beta_i(x, t), \qquad a < t < c, \ x \in \partial\Omega, \\
&v_i^-(x, a+\theta) = \chi_i^-(x, \theta) \le \chi^i(x, \theta), \qquad -\tau \le \theta \le 0, \ x \in \Omega,
\end{aligned}$$

where appropriate modifications are made when $i \in \Sigma_0$ (and hence $d_i = 0$). Our basic result is the following:

Proposition 1. *Suppose that* $v^\pm$ *and* $f^\pm$ *are as in the preceding paragraph and that* (1.2) *and* (1.5) *are satisfied with* (1.5c) *replaced by the following*:

(1.16) *if* $k \in \{1, \dots, m\}$ *and* $(t, x, \varphi) \in [a, c) \times \overline{\Omega} \times C([-\tau, 0])^m$ *with* $v^-(x, t+\theta) \le \varphi(\theta) \le v^+(x, t+\theta)$ *for all* $-\tau \le \theta \le 0$, *then*
(a) $\varphi_k(0) = v_k^+(x, t)$ *implies* $f_k(t, x, \varphi) \le f_k^+(t, x, v_t^+(x, \cdot))$, *and*
(b) $\varphi_k(0) = v_k^-(x, t)$ *implies* $f_k(t, x, \varphi) \ge f_k^-(t, x, v_t^-(x, \cdot))$.

Then (1.1) *has a unique noncontinuable mild solution* u *on* $[a, b)$ *where* $b \ge c$, *and this solution satisfies*

$$v^-(x, t) \le u(x, t) \le v^+(x, t) \quad \text{for all } (x, t) \in \overline{\Omega} \times [a, c).$$

This proposition is a fundamental result for estimating solutions to (1.1) using upper and lower solutions and differential inequalities. In Proposition 1 we (tacitly) allow the possibility that $v_k^-(x, t) = -\infty$ [and hence (1.16b) is automatically satisfied for this k] and that $v_k^+(x, t) \equiv +\infty$ [and hence (1.16a) is automatically satisfied for this k]. Note further that taking $f^\pm \equiv f$, $v_i^-(x, t) \equiv a_i$, and $v_i^+(x, t) \equiv b_i$, the method of invariant rectangles (see Remark 1.2) is a special case. This proposition is a consequence of our basic result on abstract inequalities given in §3 (see Proposition 3).

Remark 1.4. Proposition 1 also has immediate application for obtaining inequalities between solutions to (1.1). The function f is said to be *quasi-monotone* on Λ whenever the following holds:

(1.17) if $k \in \{1, \dots, m\}$ and $(t, x, \varphi), (t, x, \psi) \in [0, \infty) \times \overline{\Omega} \times C([-\tau, 0])^m$ with $\varphi(\theta), \psi(\theta) \in \Lambda$ and $\varphi(\theta) \le \psi(\theta)$ for $-\tau \le \theta \le 0$, then $\varphi_k(0) = \psi_k(0)$ implies $f_k(t, x, \varphi) \le f_k(t, x, \psi)$.

Equivalently, if φ and ψ are as in (1.17) then

$$\lim_{h \to 0+} \frac{1}{h} d(\psi(0) - \varphi(0) + h[f(t, x, \psi) - f(t, x, \varphi)]; [0, \infty)^m) = 0.$$

Observe that (1.16) automatically holds whenever $f^\pm \equiv f$ and f is quasi-monotone. In fact, if it is assumed that f^+ and f^- are quasi-monotone, then (1.16) holds whenever $f^- \le f \le f^+$ on $[0, \infty) \times \overline{\Omega} \times C([-\tau, 0])^m$.

Remark 1.5. As the proof given for Proposition 1 shows, it is not necessary to assume $v^{\pm}$ are differentiable, but merely that they are continuous on $\overline{\Omega} \times [a - \tau, c)$. In place of $(1.15)_{\pm}$, these inequalities are written in terms of integral inequalities analogous to (1.11). In particular, for $a \leq s < t < c$ assume that

$$(1.15)'_{+} \qquad v_i^{+}(\cdot, t) \geq S_i^{+}(t, s) v_i^{+}(\cdot, s) + \int_s^t T_i(t - r) B_i^{+}(r, v^{+}(\cdot, \cdot)_r)\, dr$$

and

$$(1.15)'_{-} \qquad v_i^{-}(\cdot, t) \leq S_i^{-}(t, s) v_i^{-}(\cdot, s) + \int_s^t T_i(t - r) B_i^{-}(r, v^{-}(\cdot, \cdot)_r)\, dr$$

where $S_i^{\pm}$ is defined as S_i with β_i replaced by $\beta_i^{\pm}$ and $B_i^{\pm}$ is defined as B_i with f_i replaced by $f_i^{\pm}$ [see (C4) and (C5) in §2].

Corollary 1. *Suppose that the hypotheses of Proposition* 1 *are satisfied with* $f^{\pm} \equiv f$ *and that* f *is quasi-monotone. If* $\tilde{\chi}$ *and* $\hat{\chi}$ *are members of* $\mathscr{C}([-\tau, 0])^m$ *with* $v_a^{-} \leq \tilde{\chi} \leq \hat{\chi} \leq v_a^{+}$ *and* $\tilde{u}$ *(resp.,* $\hat{u}$*) is the solution to* (1.1) *on* $\overline{\Omega} \times [a, \tilde{b})$ *(resp., on* $\overline{\Omega} \times [a, \hat{b})$*) with* $\chi = \tilde{\chi}$ *(resp.,* $\chi = \hat{\chi}$*), then* $\tilde{b} \geq \hat{b}$ *and*

$$v^{-}(x, t) \leq \tilde{u}(x, t) \leq \hat{u}(x, t) \leq v^{+}(x, t)$$

for all $(x, t) \in \overline{\Omega} \times [a, b_0)$ *where* $b_0 = \min\{c, \hat{b}\}$.

Proof. The existence of $\tilde{u}$ is a direct consequence of Proposition 1, and replacing v^{-} by $\tilde{u}$ in Proposition 1 establishes the existence of $\hat{u}$ with $\tilde{u} \leq \hat{u} \leq v^{+}$ (see Remark 1.5 and $(1.15)'_{-}$].

Remark 1.6. Note that if f is quasi-monotone and quasi-positive on $[0, \infty)^m$, then Corollary 1 always applies with $f^{\pm} = f$, $v^{-} \equiv 0$, and $v^{+} \equiv \infty$.

Results can also be obtained involving strict inequalities for solutions to (1.1). A basic assumption on the nonlinear term f implying the strict positivity of each component of nonnegative solutions to (1.1) is an irreducible type of condition and has the following form: $\Lambda \subset [0, \infty)^m$ and there is an $\overline{x} \in \Omega$ such that

(1.18) if Σ is a proper, nonempty subset of $\{1, \ldots, m\}$, $0 \leq t_1 < t_2$, and $z = (z_i)_1^m : [t_1 - \tau, t_2] \to \Lambda$ where
(a) $z_j(t) = 0$ for all $j \in \Sigma^c$ and $t_1 - \tau \leq t \leq t_2$,
(b) $z_j(t) > 0$ for all $j \in \Sigma$ and $t_1 - \tau \leq t \leq t_2$, then there is a $k \in \Sigma^c$ such that $\sup\{f_k(t, \overline{x}, z_t) : t_1 \leq t \leq s\} > 0$ for all $t_1 < s \leq t_2$.

Remark 1.7. Since $\Lambda \subset [0, \infty)^m$, it follows from the subtangential condition (1.5c) that if Σ and z are as in (1.18) and $k \in \Sigma^c$, then $f_k(t, x, z_t) \geq 0$ for all $t \in [t_1, t_2]$ and $x \in \overline{\Omega}$. Hence a crucial issue in (1.18) is that the supremum is strictly positive. In fact, property (1.18) is a direct extension of the concept of irreducibility for a quasi-positive matrix. Recall that if $A = (a_{ij})$ is an $m \times m$ (real) matrix then A is quasi-positive if $a_{ij} \geq 0$ for all $i \neq j$. Equivalently,

$$\lim_{h \to 0+} \frac{1}{h} d(\xi + hA\xi\,;\, [0, \infty)^m) = 0 \quad \text{for all } \xi = (\xi_i)_1^m \in [0, \infty)^m.$$

If $e_1, \dots, e_m$ is the natural basis for $\mathbf{R}^m$, a subspace V of $\mathbf{R}^m$ is said to be a coordinate subspace if there is a nonempty subset Σ of $\{1, \dots, m\}$ such that $\{e_i : i \in \Sigma\}$ is a basis for V. The matrix A is said to be *irreducible* if it has no proper invariant coordinate subspace. It is easy to see that a quasi-positive matrix $A = (a_{ij})$ is irreducible if and only if the following is true:

(1.19) if Σ is a proper, nonempty subset of $\{1, \dots, m\}$ and $\xi \in [0, \infty)^m$ is such that $\xi_j > 0$ for $j \in \Sigma$ and $\xi_j = 0$ for $j \in \Sigma^c$, then there is a $k \in \Sigma^c$ such that $\sum_{j=1}^m a_{kj}\xi_j > 0$ (i.e., if $\eta = A\xi$ then $\eta_k > 0$ for some $k \in \Sigma^c$).

Now suppose that $L = (L_i)_1^m$ is a bounded linear map from $C([-\tau, 0])^m$ into $\mathbf{R}^m$ that is quasi-positive

if $\varphi = (\varphi_i)_1^m \in C([-\tau, 0])^m$ with $\varphi_i \geq 0$ for all i and $\varphi_k(0) = 0$ for some k, then $L_k(\varphi) \geq 0$

(see Remark 1.1). Since L is independent of $(t, x) \in [0, \infty) \times \overline{\Omega}$, it is easy to see that (1.18) holds with $\Lambda = [0, \infty)^m$ and $f(t, x, \varphi) \equiv L(\varphi)$ only in case

(1.20) if Σ is a proper, nonempty subset of $\{1, \dots, m\}$ and $\varphi = (\varphi_i)_1^m \in C([-\tau, 0])^m$ with
(a) $\varphi_j(\theta) = 0$ for all $j \in \Sigma^c$ and $-\tau \leq \theta \leq 0$,
(b) $\varphi_j(\theta) > 0$ for all $j \in \Sigma$ and $-\tau \leq \theta \leq 0$,
then there is a $k \in \Sigma^c$ such that $L_k(\varphi) > 0$.

Identify $\mathbf{R}^m$ as a subset of $C[-\tau, 0]^m$ by defining $\hat{\xi}(\theta) \equiv \xi$ on $[-\tau, 0]$ for all $\xi \in \mathbf{R}^m$ and define the map $A = (A_i)_1^m$ from $\mathbf{R}^m$ into $\mathbf{R}^m$ by

$$A\xi = L(\hat{\xi}) \text{ and hence } A_i\xi = L_i(\hat{\xi}) \quad \text{for } i = 1, \dots, m.$$

If $e_1, \dots, e_m$ is the natural basis for $\mathbf{R}^m$ and $a_{ij} \equiv L_i(\hat{e}_j)$ for $i, j = 1, \dots, m$, then

$$A_i\xi = L_i\left[\sum_{j=1}^m \xi_j \hat{e}_j\right] = \sum_{j=1}^m \xi_j L_i(\hat{e}_j) = \sum_{j=1}^m a_{ij}\xi_j$$

and we see that A is represented by the $m \times m$ matrix $(L_i(\hat{e}_j))$. Furthermore, we have the following:

(1.21) if $A = (a_{ij})$ where $a_{ij} = L_i(\hat{e}_j)$ for $i, j = 1, \dots, m$ and $L: C([-\tau, 0])^m \to \mathbf{R}^m$ is bounded, linear, and quasi-positive, then L satisfies (1.20) if and only if A is irreducible.

It is immediate that (1.20) implies that A is irreducible. Conversely, suppose A is irreducible and φ is as in (1.20). By continuity select $\xi \in [0, \infty)^m$ such that $\xi_j = 0$ for $j \in \Sigma^c$ and $0 < \xi_j \leq \varphi_j(\theta)$ for all $-\tau \leq \theta \leq 0$ if $j \in \Sigma$. If $k \in \Sigma^c$ then $\varphi - \hat{\xi} \geq 0$ and $\varphi_k(0) - \hat{\xi}_k = 0$, so $L_k(\varphi - \hat{\xi}) \geq 0$ since L is quasi-positive. Hence $L_k(\varphi) \geq L_k(\hat{\xi}) = A_k\xi$ and it is immediate from (1.19) that L must satisfy (1.20). Therefore, the techniques presented here are related to those for functional differential equations given in Smith [17] and to reaction-diffusion systems in Martin [12].

As opposed to using (1.18) directly, we consider a more general situation which also has immediate implication for strict inequalities between two comparable solutions to (1.1). Therefore, assume that $v^{\pm}$ and $f^{\pm}$ are as in the paragraph preceding Proposition 1 and that property (1.16) in Proposition 1 is satisfied. Paralleling (1.18), we assume in addition that the following holds: there is an $\overline{x} \in \Omega$ such that

(1.22)

> if Σ is a proper, nonempty subset of $\{1, \ldots, m\}$, $0 \leq t_1 < t_2$, and $z = (z_i)_1^m : [t_1 - \tau, t_2] \to \mathbf{R}^m$ is such that
>
> (a) $z_j(t) = v_j^-(\overline{x}, t)$ for all $j \in \Sigma^c$ and $t_1 - \tau \leq t \leq t_2$,
>
> (b) $v_j^-(\overline{x}, t) < z_j(t) \leq v_j^+(\overline{x}, t)$ for all $j \in \Sigma$ and $t_1 - \tau \leq t \leq t_2$,
>
> then there is a $k \in \Sigma^c$ with
>
> $$\sup\{f_k(t, \overline{x}, z_t) - f_k^-(t, \overline{x}, v^-(\overline{x}, \cdot)) : t_1 \leq t \leq s\} > 0$$
>
> for each $t_1 < s \leq t_2$.

Notice that (1.22) is the same as (1.18) if $f^- \equiv 0$, $v^+ \equiv +\infty$, and $v^- \equiv 0$. Again observe that the crucial issue in (1.22) is that the supremum is strictly positive [compare with (1.16b)]. Our fundamental result for strict inequalities is given by the following:

Proposition 2. *Suppose the hypotheses in Proposition 1 are satisfied and also that (1.22) holds. Let $u = (u^i)_1^m$ be the mild solution to (1.1) on $[a, b]$ guaranteed by Proposition 1 and suppose in addition there is a $t_1 \in [a, a+\tau)$ such that either $u^k(\overline{x}, t_1) > v_k^-(\overline{x}, t_1)$ for some $k \in \Sigma_0$ or $u^k(x_0, t_1) > v_k^-(x_0, t_1)$ for some $k \in \Sigma_0^c$ and $x_0 \in \overline{\Omega}$. If $b > t_1 + (m-1)\tau$ then there is a $t_m \in [t_1, t_1 + (m-1)\tau]$ such that*

$$\begin{aligned} &u^i(\overline{x}, t) > v_i^-(\overline{x}, t) \quad \text{for all } i \in \Sigma_0 \text{ and } t \in (t_m, b), \\ &u^i(x, t) > v_i^-(x, t) \quad \text{for all } i \in \Sigma_0^c \text{ and all } (x, t) \in \overline{\Omega} \times (t_m, b). \end{aligned} \tag{1.23}$$

Taking $v^- \equiv 0$ and $f^- \equiv 0$ in Proposition 2 leads directly to the following:

Corollary 2. *Suppose that Σ_0 is empty (and hence $d_i > 0$ for all i), the suppositions of Proposition 1 hold with $v^- \equiv 0$, and (1.18) holds. If $u = (u^i)_1^m$ is the nonnegative mild solution to (1.1) on $[a, b)$ with $b > t_0 + (m-1)\tau$ and $u^k(x_0, t_0) > 0$ for some $(x_0, t_0) \in \Omega \times [a, a+\tau)$ and some $k \in \{1, \ldots, m\}$,* *then there is a* $t_m \in [t_0, t_0 + (m-1)\tau]$ *such that* $u^i(x, t) > 0$ *for all* $(x, t) \in \overline{\Omega} \times (t_m, b)$ *and all* $i \in \{1, \ldots, m\}$.

Now we show that Proposition 2 can be applied to obtain strict inequalities between solutions to (1.1) (compare Proposition 1 with Corollary 1). Therefore, in addition to assuming that f is quasi-monotone [see (1.17)], suppose that

there is an $\overline{x} \in \Omega$ so that if Σ is a proper nonempty subset of $\{1, \ldots, m\}$, $t_2 > t_1 \geq 0$, and $z^{\pm} = (z_i^{\pm})_1^m$ are continuous from $[t_1 - \tau, t_2]$ into Λ with

$$(1.24)\qquad \begin{aligned} &\text{(a) } z_j^+(t) = z_j^-(t) \text{ for all } i \in \Sigma^c,\ t \in [t_1 - \tau, t_2], \\ &\text{(b) } z_j^+(t) > z_j^-(t) \text{ for all } i \in \Sigma,\ t \in [t_1 - \tau, t_2], \end{aligned}$$

then there is a $k \in \Sigma^c$ such that

$$\sup\{f_k(t, \overline{x}, z_t^+) - f_k(t, \overline{x}, z_t^-);\ t_1 \leq t \leq s\} > 0$$

for each $t_1 < s \leq t_2$.

This condition is completely analogous to (1.22) and the following result is valid:

Corollary 3. *Suppose that in addition to the suppositions in Corollary* 1, Σ_0 *is empty and* (1.24) *is satisfied. If* $\tilde{u}$ *and* $\hat{u}$ *are as in Corollary* 1 *with* $b_0 > a + (m-1)\tau$ *and there is a* $(x_0, t_0) \in \Omega \times [a, a+\tau)$ *such that* $\hat{u}^k(x_0, t_0) > \tilde{u}^k(x_0, t_0)$ *for some* $k \in \{1, \ldots, m\}$, *then there is a* $t_m \in [a, a+(m-1)\tau]$ *such that* $\hat{u}^i(x, t) > \tilde{u}^i(x, t)$ *for all* $(x, t) \in \overline{\Omega} \times (t_m, b_0)$ *and all* $i \in \{1, \ldots, m\}$.

Again this is immediate from Proposition 2 by setting $v^- \equiv \tilde{u}$ and $\chi = \hat{\chi}$. A crucial property of f needed for the proof of Proposition 2 is the following:

Lemma 1.1. *Suppose that the hypotheses of Proposition* 1 *are satisfied and that* $R > 0$. *Then*

(1.25) $f_i(t, x, \varphi) - f_i^-(t, x, v_t^-(x, \cdot)) \geq -L(R)[\varphi_i(0) - v_i^-(x, t)]$ *for all* $i = 1, \ldots, m$, $(t, x) \in [0, R] \times \overline{\Omega}$, *and* $\varphi \in C([-\tau, 0])^m$ *with* $v^-(x, t+\theta) \leq \varphi(\theta) \leq v^+(x, t+\theta)$ *and* $|\varphi(\theta)| \leq R$ *for all* $-\tau \leq \theta \leq 0$

where $L(R)$ *is the Lipschitz constant for* f *in* (1.5b). *Furthermore, if* f *is quasi-monotone then*

(1.26) $f_i(t, x, \varphi) - f_i(t, x, \psi) \geq -L(R)[\varphi_i(0) - \psi_i(0)]$ *for all* $i = 1, \ldots, m$, $(t, x) \in [0, R] \times \overline{\Omega}$, *and* $\varphi, \psi \in C([-\tau, 0])^m$ *with* $v^-(x, t+\theta) \leq \varphi(\theta) \leq \psi(\theta) \leq v^+(x, t+\theta)$ *and* $|\varphi(\theta)|, |\psi(\theta)| \leq R$ *for all* $-\tau \leq \theta \leq 0$.

Proof. Let $(t, x) \in [0, R] \times \overline{\Omega}$ and let $i \in \{1, \ldots, m\}$. Define $\overline{\varphi} \in C([-\tau, 0])^m$ by $\overline{\varphi}_j \equiv \varphi_j$ for $j \neq i$ and

$$\overline{\varphi}_i(\theta) = \max\{v_i^-(x, t+\theta), \varphi_i(\theta) - \varphi_i(0) - v_i^-(x, t+\theta)\}$$

for $-\tau \le \theta \le 0$ and obsçrve that $\overline{\varphi}_i(\theta) \ge v_i^-(x, t+\theta)$, $\overline{\varphi}_i(0) = v_i^-(x, t)$, and

$$|\varphi(\theta) - \overline{\varphi}(\theta)| = |\varphi_i(\theta) - \overline{\varphi}_i(\theta)| \le \varphi_i(0) - v_i^-(x, t).$$

Therefore, by (1.5b) and (1.16b)

$$\begin{aligned} f_i(t, x, \varphi) &- f_i^-(t, x, v_t^-(x, \cdot)) \\ &= f_i(t, x, \varphi) - f_i(t, x, \overline{\varphi}) + f_i(t, x, \overline{\varphi}) - f_i^-(t, x, v_t^-(x, \cdot)) \\ &\ge f_i(t, x, \varphi) - f_i(t, x, \overline{\varphi}) \ge -L(R)\|\varphi - \overline{\varphi}\| \\ &\ge -L(R)[\varphi_i(0) - v_i^-(x, t)]. \end{aligned}$$

This establishes (1.25) and the proof of (1.26) is the same with $f^- \equiv f$ and $v(x, t+\theta) \equiv \psi(\theta)$.

Remark 1.8. Notice that under the hypotheses of Proposition 2, if $w(x, t) \equiv u(x, t) - v^-(x, t)$ on $[a - \tau, c)$, then (1.25) implies that

$$\partial_t w^i(x, t) \ge d_i \Delta w^i(x, t) - L(R) w^i(x, t), \qquad i = 1, \dots, m,$$

for R sufficiently large and $(x, t) \in \Omega \times [a-\tau, c_0]$ where $a < c_0 < c$. If $i \in \Sigma_0^c$ then the boundary conditions imply

$$\alpha_i(x) w^i(x, t) + \partial_n w^i(x, t) = \beta_i(x, t) - \beta^-(x, t) \ge 0$$

for $(x, t) \in \partial\Omega \times [a, c_0]$. Since $w^i(x, a) \ge 0$ for $x \in \overline{\Omega}$ the strong maximum principle shows the following:
(1.27)

(a) if $i \in \Sigma_0$ and $u^i(\overline{x}, t_0) > v^-(\overline{x}, t_0)$ for some $t_0 \in [a, c_0)$, then $u^i(\overline{x}, t) > v^-(\overline{x}, t)$ for all $t \in (t_0, c_0)$;

(b) if $i \in \Sigma_0^c$ and $u^i(x_0, t_0) > v^-(x_0, t_0)$ for some$(x_0, t_0) \in \Omega \times [a, c_0)$, then $u^i(x, t) > v^-(x, t)$ for all $(x, t) \in \overline{\Omega} \times (t_0, c_0)$.

Remark 1.9. There is a convenient criterion to check if (1.24) is satisfied using the Fréchet derivative of the map $\psi \to f(t, \overline{x}, \psi)$. In particular, if $f(t, \overline{x}, \cdot)$ is continuously Fréchet differentiable for each $t > a$ and the Fréchet derivative of $f(t, \overline{x}, \cdot)$ at φ satisfies (1.20) [i.e., is irreducible—see (1.21)] for all but an at most countable number of (t, φ) in $[a, \infty) \times C([-\tau, 0])^m$, then f also satisfies (1.24) (the proof follows from Lemma 3.2 in the next section).

Remark 1.10. Since the boundary conditions associated with diffusion in this system always contain the normal derivative [see (1.4)], we are able to use the space of continuous functions $C(\overline{\Omega})$ in writing our system as an abstract integral equation [see (1.11)]. If the boundary condition for the ith equation is homogeneous and Dirichlet [$u^i(x, t) = 0$ for $t > a$ and $x \in \partial\Omega$], then our techniques still apply in essentially the same manner except that $C(\overline{\Omega})_0 \equiv \{\varphi \in C(\overline{\Omega}): \varphi(x) = 0$ for all $x \in \partial\Omega\}$ must be used as the underlying space for the ith component u^i. That is, in place of the space $C(\overline{\Omega})^m$ use $X = \prod_{i=1}^m X_i$

where $X_i = C(\overline{\Omega})_0$ if the boundary condition for u^i is homogeneous and Dirichlet and $X_i = C(\overline{\Omega})$ otherwise. All of the preceding results remain valid except that strict inequalities hold only for $x \in \Omega$ (as opposed to $x \in \overline{\Omega}$) for all jth components with $X_j = C(\overline{\Omega})_0$. Furthermore, the space $\mathscr{C}$ is taken to be $\mathscr{C}([-\tau, 0], X)$ and it must be assumed that B defined by (1.10) maps $\mathscr{C}_\Lambda$ into $\mathscr{C}$ [i.e., if $X_j = C(\overline{\Omega})_0$ and $\varphi \in \mathscr{C}_\Lambda$, then $f_j(t, x, \varphi(x, \cdot)) = 0$ for all $t \geq 0$ and $x \in \partial\Omega$]. The difficulty arising under Dirichlet boundary conditions is caused by the fact that the Laplacian operator with homogeneous Dirichlet boundary conditions is not densely defined in the space $C(\overline{\Omega})$. This problem can also be circumvented using the space $L^p(\Omega)$ where $1 \leq p < \infty$ [in fact, nonhomogeneous Dirichlet boundary conditions can also be handled using $L^p(\Omega)$]. However, in this situation additional assumptions must be made on the functional nonlinearity f, and so it is not described here.

2. Abstract results

Let X be a real or complex Banach space with norm denoted $|\cdot|$, let τ be a positive number, and denote by $\mathscr{C} = \mathscr{C}([-\tau, 0]; X)$ the space of all continuous functions $\varphi: [-\tau, 0] \to X$ with $\|\varphi\| \equiv \max\{|\varphi(\theta)|: -\tau \leq \theta \leq 0\}$. The purpose of this section is to establish fundamental results for the existence and behavior of solutions to a class of abstract semilinear integral equations involving functional nonlinearities, and then apply these results to the equations in §1. Many of these ideas are related to those in Lightbourne [8] and Martin [10].

Suppose that a is a real number and $T = \{T(t, s): t \geq s \geq a\}$ is a family of bounded linear operators from X into X that satisfy

(T1) $T(t, t)x \equiv x$ and $T(t, s)T(s, r)X \equiv T(t, r)x$ for all $t \geq s \geq r \geq a$.

(T2) For each $x \in X$ the map $(t, s) \to T(t, s)x$ is continuous for $t \geq s \geq a$.

(T3) There are numbers $\hat{M} \geq 1$ and $\omega \in \mathbf{R}$ such that $\|T(t, s)\| \equiv \sup\{|T(t, s)x|: |x| \leq 1\} \leq \hat{M}e^{\omega(t-s)}$ for all $t \geq s \geq a$.

Such a family T is a C_0 linear evolution system, and if $a = 0$ and $T(t, s) \equiv T(t - s)$ for $t \geq s \geq 0$, then T is a C_0 linear semigroup. In addition to the family T, we also consider a companion family $S = \{S(t, s): t \geq s \geq a\}$ having the following two properties:

(S1) $t \to S(t, a)0$ is continuous from $[a, \infty)$ into X (where 0 is the zero of X).

(S2) $S(t, r)x + T(t, s)y = S(t, s)[S(s, r)x + y]$ for all $x, y \in X$ and $t \geq s \geq r \geq a$.

Note in particular that by setting $y = 0$ in (S2) we have the evolution property $S(t, r)x = S(t, s)S(s, r)x$ for $t \geq s \geq r \geq a$. Equivalently, it may be assumed that there is a continuous function $\hat{\mu}: [a, \infty] \to X$ such that

(S3) $S(t, s)x = T(t, s)[x - \hat{\mu}(s)] + \hat{\mu}(t)$ for all $x \in X$ and $t \geq s \geq a$.

Therefore, the family S consists of affine operators on X and corresponds to solutions of linear differential equations having nonhomogeneous terms.

Remark 2.1. It is immediate to verify that (S3) implies that (S1) and (S2) hold whenever $\hat{\mu}$ is continuous. To see that (S1) and (S2) imply (S3), set $\hat{\mu}(t) \equiv S(t, a)0$ for $t \geq a$, and then for each $z \in X$ and $t \geq s \geq a$, set $x = 0$, $r = a$, and $y = z - S(s, a)0$ in (S2) to obtain that

$$\begin{aligned} T(t, s)[z - \hat{\mu}(s)] + \hat{\mu}(t) &= T(t, s)[z - S(s, a)0] + S(t, a)0 \\ &= S(t, s)[S(s, a)0 + \{z - S(s, a)0\}] = S(t, s)z. \end{aligned}$$

Thus (S1) and (S2) imply (S3).

It is assumed throughout this section that the following hypotheses are satisfied:

(H1) D is a closed subset of $[a - \tau, \infty) \times X$ and $D(t) \equiv \{x \in X : (t, x) \in D\}$ is nonempty for each $t \geq a - \tau$.

(H2) $\mathscr{D}$ is the closed subset of $[a, \infty) \times \mathscr{C}$ defined by $\mathscr{D} \equiv \{(t, \varphi) : \varphi(\theta) \in D(t + \theta)$ for all $-\tau \leq \theta \leq 0\}$. Also, $\mathscr{D}(t) \equiv \{\varphi \in \mathscr{C} : (t, \varphi) \in \mathscr{D}\}$ for each $t \geq a$, and we assume that $\mathscr{D}(t)$ is nonempty for each set $t \geq a$.

(H3) For each $b > a$ there are a $\hat{K}(b) > 0$ and a continuous nondecreasing function $\eta_b : [0, b - a) \to [0, \infty)$ satisfying $\eta_b(0) = 0$ with the property that if $a \leq t_1 < t_2 \leq b$, $x_1 \in D(t_1)$, and $x_2 \in D(t_2)$, then there is a continuous function $w : [t_1, t_2] \to X$ such that $w(t_1) = x_1$, $w(t_2) = x_2$, $w(t) \in D(t)$ for $t_1 < t < t_2$, and

$$|w(t) - w(s)| \leq \eta_b(|t - s|) + \hat{K}(b)|t - s|\frac{|x_2 - x_1|}{t_2 - t_1}$$

for all $s, t \in [t_1, t_2]$.

(H4) B is continuous from $D(B)$ into X where $\mathscr{D} \subset D(B) \subset [a, \infty) \times \mathscr{C}$.

Remark 2.2. Note that (H1) and (H3) imply that $\mathscr{D}(t)$ is nonempty for $t \geq a + \tau$, for if $x_1 \in D(t - \tau)$, $x_2 \in D(t)$, and w is as in (H3) with $t_1 = t - \tau$ and $t_2 = t$, then $(t, w) \in \mathscr{D}$. Furthermore, if D is convex then (H3) is automatically satisfied by defining

$$w(t) = \frac{(t_2 - t)x_1 + (t - t_1)x_2}{(t_2 - t_1)} \quad \text{for } t_1 \leq t \leq t_2.$$

Since D is convex,

$$(t, w(t)) = \frac{(t_2 - t)}{t_2 - t_1}(t_1, x_1) + \frac{(t - t_1)}{(t_2 - t_1)}(t_2, x_2) \in D$$

and since

$$|w(t) - w(s)| = \left|\frac{(s - t)x_1 + (t - s)x_2}{(t_2 - t_1)}\right| \leq \frac{|x_1 - x_2|}{t_2 - t_1}|t - s|$$

for $t_1 \leq s < t \leq t_2$, we see that (H3) is satisfied with $\hat{K}(b) \equiv 1$ and $\eta_b = 0$ if D is convex. However, in order to apply our results to differential inequalities and

Lyapunov techniques, we cannot assume that D is convex and so the technical hypothesis (H3) seems necessary for the abstract techniques to have a wide range of applicability. Furthermore, an example given in Leela and Moauro [6] shows that invariance criteria of the type presented here are not valid for general closed sets D, even when $X \equiv \mathbf{R}$. A further example involving a pathwise connected set in the plane is given at the end of this section.

If $b > a$ and u is a continuous function from $[a-\tau, b]$ into X and $t \in [a, b]$, then u_t denotes the member of $\mathscr{C}$ defined by $u_t(\theta) = u(t+\theta)$ for $-\tau \le \theta \le 0$. Assuming (H1)–(H4), we consider the abstract integral equation

$$\text{(2.1)} \qquad \begin{aligned} u(t) &= S(t, a)\chi(0) + \int_a^t T(t, r)B(r, u_r)\,dr, \qquad a \le t < b, \\ u(a+\theta) &= \chi(\theta) \quad \text{for } -\tau \le \theta \le 0, \end{aligned}$$

where $\chi \in \mathscr{D}(a)$ is given. A function $u: [a-\tau, b) \to X$ is a solution to (2.1) if u is continuous, $u_a = \chi$, $(t, u_t) \in \mathscr{D}$ for all $t \in [a, b)$, and u satisfies the first equation in (2.1). Note in particular that $(t, u_t) \in \mathscr{D}$ for all $t \in [a, b)$ only in case $u(t) \in D(t)$ for all $t \in [a, b)$, and hence existence results for (2.1) also give criteria for invariant sets. In fact, with the notation

$$d(x; D(t)) \equiv \inf\{|x-y| : y \in D(t)\} \quad \text{for } x \in X,\ t \ge a,$$

the fundamental criterion for the invariance of the set $\mathscr{D}$ is given by

(2.2)

$$\lim_{h\to 0+} \frac{1}{h} d\left(S(t+h, t)\varphi(0) + \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr;\ D(t+h)\right) = 0$$

$$\text{for } (t, \varphi) \in \mathscr{D}.$$

This type of "subtangential condition" is crucial to our analysis and has been used frequently in connection with ordinary differential systems as well as functional differential systems (see, e.g., [8, 10, 16]). Observe that if (2.1) has a solution u on $[a, b]$ then

$$\begin{aligned} &\frac{1}{h} d\left(S(a+h, a)\chi(0) + \int_a^{a+h} T(a+h, r)B(a, \chi)\,dr;\ D(a+h)\right) \\ &\quad \le \frac{1}{h}\left|S(a+h, a)\chi(0) + \int_a^{a+h} T(a+h, r)B(a, \chi)\,dr - u(a+h)\right| \\ &\quad = \frac{1}{h}\left|\int_a^{a+h} T(a+h, r)B(a, \chi)\,dr - \int_a^{a+h} T(a+h, r)B(r, u_r)\,dr\right| \\ &\quad \le \frac{1}{h}\int_a^{a+h} \hat{M}e^{\omega h}|B(a, \chi) - B(r, u_r)|\,dr \\ &\quad \to 0 \quad \text{as } h \to 0+ \end{aligned}$$

since B is continuous and $u_r \to \chi$ as $r \to a+$. Thus, if (2.1) has a local solution with (a, χ) replaced by $(\overline{a}, \varphi)$ for every $(\overline{a}, \varphi) \in \mathscr{D}$, then (2.2) must be valid and hence is a necessary condition for local existence of solutions.

We now state our main existence result for (2.1) and we assume that (T1)–(T3), (S1)–(S3), and (H1)–(H4) are always valid.

Theorem 2. *Suppose that* (2.2) *holds and for each* $R > 0$ *there are an* $L_R > 0$ *and a continuous* $\nu_R : [0, \infty) \to [0, \infty)$ *such that* $\nu_R(0) = 0$ *and*

$$\tag{2.3} \begin{aligned} &|B(t, \varphi) - B(s, \psi)| \le \nu_R(|t - s|) + L_R\|\varphi - \psi\| \text{ for all } (t, \varphi), \\ &(s, \psi) \in \mathscr{D} \text{ with } \|\varphi\|, \|\psi\| \le R \text{ and } a \le s, t \le a + R. \end{aligned}$$

Then (2.1) *has a unique noncontinuable solution* u *on an interval of the form* $[a, b)$ *where* $a < b \le +\infty$. *Moreover,* $u(t) \in D(t)$ *for* $a \le t < b$ *and if* $b < +\infty$ *then* $\|u_t\| \to \infty$ *as* $t \to b^-$.

A detailed proof of this theorem is given in the last section of this paper. Notice that if the hypotheses in Theorem 2 hold for some a, then they hold with a replaced by $\bar{a}$ for any $\bar{a} \ge a$. Thus these local existence results can be combined with standard continuation arguments in order to obtain solutions defined on a maximal interval. Therefore we prove only the local existence of solutions.

As a consequence of this invariance criterion we add the following corollary to these results

Corollary 4. *Suppose* K *is a closed, convex subset of* X *and* (T1)–(T3), (S1)–(S3), *and* (H1)–(H4) *are satisfied with* $D(t) \equiv K$ *for all* $t \ge a$. *Suppose further that* (2.3) *holds and*

$$\tag{2.4} \begin{aligned} &\text{(a) } S(t, s): K \to K \text{ for } t \ge s \ge a \text{ and} \\ &\text{(b) } \lim (1/h) d(\varphi(0) + hB(t, \varphi); K) = 0 \text{ for } (t, \varphi) \in \mathscr{D}. \end{aligned}$$

Then (2.1) *has a unique noncontinuable solution* u *on* $[a, b)$ *from some* $b > a$ *and* $u(t) \in K$ *for all* $a - \tau \le t < b$.

Proof. The function $h \to d(x + hy; K)$ is convex (since K is convex) and so it follows from (2.4) that

$$(1/h) d(\varphi(0) + hB(t, \varphi); K) \downarrow 0 \quad \text{as } h \downarrow 0.$$

Using this fact along with the continuity of B shows further that if $\mathscr{E}$ is a compact subset of $\mathscr{D}$ then

$$\lim_{h \to 0+} \frac{1}{h} d(\varphi(0) + hB(t, \varphi); K) = 0$$

uniformly for $(t, \varphi) \in \mathscr{E}$. So let $(t, \varphi) \in \mathscr{D}$ be given and set $\psi_h(\theta) = S(t + h, t)\varphi(\theta)$ for $h \ge 0$ and $-\tau \le \theta \le 0$. Then $\{(t, \psi_h) : 0 \le h \le 1\}$ is a compact subset of $\mathscr{D}$ and it follows that

$$\begin{aligned} &\frac{1}{h} d\left(S(t + h, t)\varphi(0) + \int_t^{t+h} T(t + h - r, r)B(t, \varphi)\, dr;\ D(t + h)\right) \\ &\le \frac{1}{h} d(S(t + h, t)\varphi(0) + hB(t, \varphi); K) + \varepsilon_1(h) \\ &\le \frac{1}{h} d(\psi_h(0) + hB(t, \psi_h); K) + \varepsilon_2(h) \\ &\to 0 \quad \text{as } h \to 0+. \end{aligned}$$

This shows that (2.2) is fulfilled and hence the corollary is a consequence of Theorem 2.

We now show how the results on invariant sets can be applied to obtain inequalities for solutions. Suppose that X_+ is a closed cone in X (i.e., $x, y \in X_+$, $\alpha \geq 0$ implies $x+y$, $\alpha x \in X_+$) with the property that $x, -x \in X_+ \Leftrightarrow x = 0$. Define the partial ordering "$\geq$" on X by $x \geq y$ only in case $x - y \in X_+$ (and hence $X_+ = \{x \in X : x \geq 0\}$). It is also assumed that X with this ordering is a vector lattice:

for each $x, y \in X$, $z = \sup\{x, y\}$ exists—that is, $z \geq x$, $z \geq y$, and if $z_1 \geq x$, $z_1 \geq y$ then $z \leq z_1$.

Throughout this section we use the notation

$$\begin{aligned} x \vee y &= \sup\{x, y\}, \qquad x \wedge y = \inf\{x, y\} \equiv -\sup\{-x, -y\}, \\ x_+ &= x \vee 0, \qquad x_- = -(x \wedge 0), \quad \text{and} \quad |x|_+ = x_+ + x_-. \end{aligned} \tag{2.5}$$

Observe that $x_- = (-x) \vee 0 = (-x)_+$, $|x_+|_+ \leq |x|_+$, $|x_-|_+ \leq |x|_+$, and $|x|_+ = |-x|_+$. All of the assertions concerning lattices made here can be found in Vulikh [20]. In particular, the following properties are valid:

(2.6)
(a) $x \vee y = (x - y)_+ + y$ (see [20, p. 49]);
(b) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ (see [20, p. 55]);
(c) $x = x_+ - x_-$ and if $y, z \geq 0$ with $x = y - z$, then $y \geq x_+$, $z \geq x_-$ (see [20, p. 53]).

Furthermore, we always assume X is a Banach lattice:

$$|x|_+ \leq |y|_+ \text{ implies } |x| \leq |y| \text{ for all } x, y \in X. \tag{2.7}$$

That is, the norm $|\cdot|$ on X is monotonic with respect to the ordering $\leq$. Also, from [20, pp. 146, 174],

(2.8)
(a) $|(x \vee z) - (y \vee z)| \leq |x - y|$ for $x, y, z \in X$;
(b) $|(x \wedge z) - (y \wedge z)| \leq |x - y|$ for $x, y, z \in X$.

Of particular interest in these considerations are order intervals, and so if $w, z \in X$ with $w \leq z$ define

$$[w, +\infty) = \{x \in X : x \geq w\}, \qquad (-\infty, z] = \{x \in X : x \leq z\}$$
$$\text{and} \quad [w, z] = \{x \in X : w \leq x \leq z\}.$$

Also, for simplicity of notation we allow w to be $-\infty$ and z to be $+\infty$ in this notation. Therefore,

$$[-\infty, +\infty) \equiv (-\infty, \infty] \equiv X, \qquad [-\infty, z] \equiv (-\infty, z],$$
$$[w, +\infty] \equiv [w, +\infty).$$

For the remainder of this section it is assumed that v^- and v^+ are continuous functions from $[a - \tau, b)$ into X such that $v^-(t) \leq v^+(t)$ for all

$t \in [a-\tau, b)$. Our purpose is to develop criteria for the solution u to (2.1) to satisfy $v^-(t) \le u(t) \le v^+(t)$ for all $t \in [a-\tau, b)$ {i.e., $u(t) \in [v^-(t), v^+(t)]$ for all $t \in [a-\tau, b]$}. We also allow the possibility that $v^-(t) \equiv -\infty$ or that $v^+(t) \equiv +\infty$, and hence our results also include one-sided estimates for solutions. Observe that if we take $D(t) = [v^-(t), v^+(t)]$ for $a-\tau \le t < b$, then Theorem 2 may be applied directly to determine if $v^-(t) \le u(t) \le v^+(t)$ [i.e., $u(t) \in D(t)$] for $a-\tau \le t < b$. Therefore, the techniques of this section are essentially the verification of the hypotheses in Theorem 2 with $D(t) = [v^-(t), v^+(t)]$ for $a-\tau \le t < b$.

We continue to suppose (T1)–(T3) and (S1)–(S3) are satisfied and, additionally, we suppose T is positive:

(T4) $$T(t, s): X_+ \to X_+ \quad \text{for } t \ge s \ge a.$$

Let E be a subset of $[a-\tau, \infty) \times X$ such that $E(t) \equiv \{x \in X: (t, x) \in E\}$ is nonempty for all t and define

$$\mathscr{E} = \{(t, \varphi) \in [a, \infty) \times \mathscr{C}: (t+\theta, \varphi(\theta)) \in E \quad \text{for } -\tau \le \theta \le 0\}.$$

Assume now that B is a continuous function from $\mathscr{E}$ into X and continue to consider the integral equation (2.1) where $(a, \chi) \in \mathscr{E}$ and $v^-(a+\theta) \le \chi(\theta) \le v^+(a+\theta)$ for $-\tau \le \theta \le 0$. The aim now is to place conditions on S, B, and $v^\pm$ to ensure that $v^-(t) \le u(t) \le v^+(t)$ for $a \le t \le c$. First assume the following:

(C1) S^+ and S^- satisfy (S1)–(S2) with S replaced by S^+ and S^-, respectively.

(C2) $S^-(t, s)x \le S(t, s)x \le S^+(t, s)x$ for all $t \ge s \ge a$ and $x \in X$.

(C3) $[v^-(t), v^+(t)] \subset E(t)$ for all $a-\tau \le t \le b$ and, since $v^\pm$ are continuous, for each $c > a$ let $\overline{\nu}_c: [0, c-a] \to [0, \infty)$ be continuous and increasing with $\overline{\nu}_c(0) = 0$ and

$$|v^\pm(t) - v^\pm(s)| \le \hat{\nu}_c(|t-s|) \quad \text{for all } a \le t,\ s \le c.$$

Furthermore, suppose that B^+ and B^- are continuous functions from $\mathscr{E}$ into X and that v^+ and v^- satisfy the following integral inequalities:

(C4) $v^+(t+h) \ge S^+(t+h, t)v^+(t) + \int_t^{t+h} T(t+h, r)B^+(r, v_r^+)\, dr$ for $a \le t < t+h < b$.

(C5) $v^-(t+h) \le S^-(t+h, t)v^-(t) + \int_t^{t+h} T(t+h, r)B^-(r, v_r^-)\, dr$ for $a \le t < t+h < b$.

Remark 2.3. Note that if

$$v^+(t) = S^+(t, a)v^+(a) + \int_a^t T(t, r)B^+(r, v_r^+)\, dr \quad \text{for } a \le t < b$$

then (C4) is automatically satisfied. To see that this is so, apply (S2) and (T1) to obtain

$$\begin{aligned} v^+(t+h) &= S^+(t+h\,,\,a)v^+(a) + T(t+h\,,\,t)\int_a^t T(t\,,\,r)B^+(r\,,\,v_r^+)\,dr \\ &\quad + \int_t^{t+h} T(t+h\,,\,r)B^+(r\,,\,v_r^+)\,dr \\ &= S(t+h\,,\,t)\left[S(t\,,\,a)v^+(a) + \int_a^t T(t\,,\,r)B^+(r\,,\,v_r^+)\,dr\right] \\ &\quad + \int_t^{t+h} T(t+h\,,\,r)B^+(r\,,\,v_r^+)\,dr \\ &= S(t+h\,,\,t)v^+(t) + \int_t^{t+h} T(t+h\,,\,r)B^+(r\,,\,v_r^+)\,dr. \end{aligned}$$

Hence (C4) holds with $\geq$ replaced by $=$. Similarly, if

$$v^-(t) = S^-(t\,,\,a)v^-(a) + \int_a^t T(t\,,\,r)B^-(r\,,\,v_r^-)\,dr \quad \text{for } a \leq t \leq b\,,$$

then (C5) is valid. In particular, taking $S^{\pm} = S$ and $B^{\pm} = B$, (C4) and (C5) are valid if $v^{\pm}$ are solutions to the integral equation (1.2) or (2.5). Also, we tacitly assume (C3) and (C4) [respectively, (C5)] are automatically fulfilled if $v^+(t) \equiv +\infty$ [respectively, $v^-(t) \equiv -\infty$].

Although we only need to require B^+ and B^- to be continuous, we need enough conditions on B and T to ensure local existence, and so we assume that

(C6) B satisfies the Lipschitz condition (2.3) with $\mathscr{D}$ replaced by $\mathscr{E}$.

Under these hypotheses we have the following:

Proposition 3. *In addition to* (C1)-(C6) *suppose that*

(2.9) $\lim_{h\to 0+} \frac{1}{h} d(v^+(t) - \varphi(0) + h[B^+(t\,,\,v_t^+) - B(t\,,\,\varphi)]\,;\, X_+) = 0$ *for all* $a \leq t < b$ *and* $(t\,,\,\varphi) \in \mathscr{E}$ *with* $v^-(t+\theta) \leq \varphi(\theta) \leq v^+(t+\theta)$ *for* $-\tau \leq \theta \leq 0$

and

(2.10) $\lim_{h\to 0+} \frac{1}{h} d(\varphi(0) - v^-(t) + h[B(t\,,\,\varphi) - B^-(t\,,\,v_t^-)]\,;\, X_+) = 0$ *for all* $a \leq t < b$ *and* $(t\,,\,\varphi) \in \mathscr{E}$ *with* $v^-(t+\theta) \leq \varphi(\theta) \leq v^+(t+\theta)$ *for* $-\tau \leq \theta \leq 0$.

If $v^-(a+\theta) \leq \chi(\theta) \leq v^+(a+\theta)$ *for* $-\tau \leq \theta \leq 0$, *then* (2.1) *has a solution* u *on* $[a\,,\,\overline{b}]$ *for some* $a < \overline{b} \leq b$ *such that*

$$v^-(t) \leq u(t) \leq v^+(t) \quad \text{for all } t \in [a\,,\,\overline{b}). \tag{2.11}$$

Remark 2.4. If $v^+(t) \equiv +\infty$ then (2.9) is assumed to be automatically satisfied and if $v^-(t) \equiv -\infty$ then (2.10) is assumed satisfied. For example, if $v^+(t) \equiv +\infty$ and $v^-(t) \equiv 0$, then $[v^-(t)\,,\,v^+(t)] = X_+$. Assuming further that

$S(t,s): X_+ \to X_+$ for all $t \geq s \geq a$, we see that (C5) holds if $S^- = S$ and B^- is defined by $B^-(t,\varphi) = B(t,\varphi) - B(t,0)$. Hypothesis (2.10) then becomes

$$\lim_{h \to 0+} \frac{1}{h} d(\varphi(0) + hB(t,\varphi); X_+) = 0 \tag{2.12}$$

whenever $(t,\varphi) \in [a,\infty) \times \mathscr{C}$ with $\varphi(\theta) \geq 0$ for all $-\tau \leq \theta \leq 0$. (Notice that this result reduces to Corollary 1 in §1 with $K = X_+$.) The function B is said to be *quasi-positive* (on X_+) when (2.12) holds.

For the proof of Proposition 1 we use two preliminary lemmas.

Lemma 2.1. *Suppose that* $w, z \in X$ *with* $w \leq z$. *then*

(i) $d(x;[w,\infty)) = |(x-w)_-| = d(x-w; X_+)$,
(ii) $d(x;(-\infty,z]) = |(x-z)_+| = d(z-x; X_+)$,
(iii) $d(x,[w,z]) \leq d(x;(-\infty,z]) + d(x;[w,\infty))$

for each $x \in X$.

Proof. Parts (i) and (ii) will follow routinely once it is shown that

$$d(x; X_+) = |x - x_+| = |x_-| \quad \text{for all } x \in X.$$

So let $x \in X$ and note that if $y \in X_+$ then

$$x - [y + (x-y)_+] = (x-y) - (x-y)_+ = -(x-y)_-$$

and hence

$$|x - [y + (x-y)_+]|_+ = |-(x-y)_-|_+ = |(x-y)_-|_+ \leq |x-y|_+.$$

Therefore, $|x - [y + (x-y)_+]| \leq |x-y|$ by norm monotonicity [see (2.7)] and so

$$|x-y| \geq |x - [y + (x-y)_+]| = |-(x-y)_-|.$$

However, $x = [y + (x-y)_+] - (x-y)_-$ and the second part of (2.6c) implies $(x-y)_- \geq x_-$. Since $x_- \geq 0$, the monotonicity of the norm shows that

$$|x_-| \leq |(x-y)_-| = |x - [y + (x-y)_+]| \leq |x-y|$$

and hence

$$|x-y| \geq |x_-| = |x - x_+| \quad \text{for all } y \in X_+.$$

This shows $d(x; X_+) = |x_-|$ since $x_+ \in X_+$, and it follows that (i) and (ii) are true. To prove (iii), set $x = -z$ and $y = -x$ in (2.6a), which shows that

$$(-z) \vee (-x) = (-z + x)_+ - x$$

and hence $x - x \wedge z = (x-z)_+$. Thus if $x \geq w$ then $x \wedge z \in [w,z]$ and

$$d(x;(-\infty,z]) = |(x-z)_+| = |x - (x \wedge z)| \geq d(x;[w,z]).$$

Clearly, $d(x;(-\infty,z)) \leq d(x;[w,z])$ and so

$$d(x;(-\infty,z]) = d(x;[w,z]) = |x - (x \wedge z)| \quad \text{if } x \geq w. \tag{2.13}$$

Since $|d(x;[w,z]) - d(y;[w,z])| \leq |x-y|$, we see that

$$d(x;[w,z]) \leq d(w \vee x;[w,z]) + |x - (w \vee x)|.$$

But $w \vee x \geq w$, so using (2.13), (2.6b), and (2.8a),

$$\begin{aligned} d(x; [w, z]) &\leq |(w \vee x) - (w \vee x) \wedge z| + |x - (w \vee x)| \\ &= |(w \vee x) - (w \wedge z) \vee (x \wedge z)| + |x - (w \vee x)| \\ &= |(w \vee x) - w \vee (x \wedge z)| + |x - (w \vee x)| \\ &\leq |x - (x \wedge z)| + |x - (w \vee x)|. \end{aligned}$$

Setting $x = -z$ and $y = -x$ in (2.6a) shows $x = (x \wedge z) = (x - z)_+$, and setting $x = w$ and $y = x$ in (2.6a) shows $x - (w \vee x) = (x - x)_+$. But $|(w - x)_+| = |(x - w)_-|$ and we see from parts (i) and (ii) that (iii) is also true.

Lemma 2.2. *Suppose that $v^{\pm}$ satisfy* (C3) *and define $D(t) = [v^-(t), v^+(t)]$ for all $t \in [a - \tau, b]$. Then hypothesis* (H3) *in* §1 *is satisfied on $[a, b]$.*

Proof. Suppose $a \leq t_1 < t_2 \leq b$ and $x_j \in [v^-(t_j), v^+(t_j)]$ for $j = 1, 2$. Define

$$L(t) = \frac{(t_2 - t_1)x_1 + (t - t_1)x_2}{t_2 - t_1} \quad \text{for } t_1 \leq t \leq t_2$$

and note that

$$(2.14) \qquad |L(t) - L(s)| = |t - s| \cdot |t_2 - t_1|^{-1} |x_2 - x_1| \quad \text{for } s, t \in [t_1, t_2].$$

Setting $w(t) = [L(t) \wedge v^+(t)] \vee v^-(t)$ for $t_1 \leq t \leq t_2$, we see that $w(t) \in [v^-(t), v^+(t)] = D(t)$ and, by (2.14), (C3), and the inequalities in (2.8), it follows that

$$\begin{aligned} |w(t) - w(s)| &\leq |[L(t) \wedge v^+(t)] \vee v^-(t) - [L(t) \wedge v^+(t)] \vee v^-(s)| \\ &\quad + |[L(t) \wedge v^+(t)] \vee v^-(s) - [L(t) \wedge v^+(s)] \vee v^-(s)| \\ &\leq |v^-(t) - v^-(s)| + |L(t) \wedge v^+(t) - L(s) \wedge v^+(s)| \\ &\leq \hat{\nu}_c(|t - s|) + |L(t) \wedge v^+(t) - L(t) \wedge v^+(s)| \\ &\quad + |L(t) \wedge v^+(s) - L(s) \wedge v^+(s)| \\ &\leq \hat{\nu}_c(|t - s|) + |v^+(t) - v^+(s)| + |L(t) - L(s)| \\ &\leq 2\overline{\nu}_c(|t - s|) + |x_2 - x_1| \cdot (t_2 - t_1)^{-1} |t - s|. \end{aligned}$$

Thus (H3) is valid on $[a, c]$ with $\hat{K}(c) = 1$ and $\eta_c = 2\overline{\nu}_c$.

Proof of Proposition 3. This result will follow from Theorem 2 once it is shown that the subtangential condition (2.2) is satisfied on $[a, c]$ with $D(t) = [v^-(t), v^+(t)]$. First note that

$$(2.15) \qquad d(x + y; X_+) \leq d(x; X_+) \quad \text{for all } x \in X,\ y \in X_+.$$

Also,if $x \in X$ and $t \geq s \geq a$ then, by (T3) and (T4),

$$\begin{aligned} d(T(t, s)x; X_+) &\leq |T(t, s)x - T(t, s)y| + d(T(t, s)y; X_+) \\ &= |T(t, s)(x - y)| \leq \hat{M} e^{\omega(t-s)} |x - y| \end{aligned}$$

for all $y \in X_+$. Therefore

$$d(T(t,s)x; X_+) \le \hat{M}e^{\omega(t-s)}d(x; X_+) \quad \text{for } t \ge s \ge a \text{ and } x \in X. \tag{2.16}$$

Now let $(t, \varphi) \in \mathscr{D}$. By (2.6b), (C4), and (2.15) it follows that

$$\begin{aligned} d&\left(S(t+h, t)\varphi(0) + \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr; (-\infty, v^+(t+h)]\right) \\ &= d\left(v^+(t+h) - S(t+h, t)\varphi(0) - \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr; X_+\right) \\ &\le d\Big(S^+(t+h, t)v^+(t) - S(t+h, t)\varphi(0) \\ &\qquad + \int_t^{t+h} T(t+h, r)[B^+(r, v_r^+) - B(t, \varphi)]\,dr; X_+\Big). \end{aligned}$$

However, (C2) and the properties of S imply

$$\begin{aligned} S^+(t+h, t)v^+(t) - S(t+h, t)\varphi(0) &\ge S(t+h, t)v^+(t) - S(t+h, t)\varphi(0) \\ &= T(t+h, t)[v^+(t) - \varphi(0)] \end{aligned}$$

and the continuity of T, B, and B^+ imply

$$\int_t^{t+h} T(t+h, r)[B^+(r, v_r^+) - B(t, \varphi)]\,dr = hT(t+h, t)[B^+(t, v_t^+) - B(t, \varphi)] + o(h)$$

where $h^{-1}|o(h)| \to 0$ as $h \to 0+$. By (2.16) and (2.9),

$$\begin{aligned} d&\left(S(t+h, t)\varphi(0) + \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr; (-\infty, v^+(t+h)]\right) \\ &\le d(T(t+h, t)[v^+(t) - \varphi(0)) + h(B^+(t, v_t^+) - B(t, \varphi)]; X_+) + |o(h)| \\ &\le \hat{M}e^{\omega h}d(v^+(t) - \varphi(0) + h(B^+(t, v_t^+) - B(t, \varphi)); X_+) + |o(h)| \end{aligned}$$

and it follows that

$$\lim_{h\to 0+} \frac{1}{h} d\left(S(t+h, t)\varphi(0) + \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr; (-\infty, v^+(t+h)]\right) = 0. \tag{2.17}$$

In a similar manner it also follows that

$$\lim_{h\to 0+} \frac{1}{h} d\left(S(t+h, t)\varphi(0) + \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr; [v^-(t+h), +\infty)\right) = 0. \tag{2.18}$$

Consequently, (2.17) and (2.18) along with (iii) in Lemma 2.1 show that

$$\begin{aligned} \lim_{h\to 0+} \frac{1}{h} d\Big(S(t+h, t)\varphi(0) &+ \int_t^{t+h} T(t+h, r)B(t, \varphi)\,dr; \\ &[v^-(t+h), v^+(t+h)]\Big) = 0, \end{aligned}$$

which is precisely (2.2) in this case. This completes the proof.

A function B from $\mathscr{E}$ into X is said to be *quasi-monotone* (on $\mathscr{E}$ relative to X_+) if

(2.19) $\lim_{h\to 0+} \frac{1}{h} d(\psi(0) - \varphi(0) + h[B(t, \psi) - B(t, \varphi)]; X_+) = 0$ for all (t, ψ), $(t, \varphi) \in \mathscr{E}$ with $\varphi \leq \psi$.

Notice in particular that if B is quasi-monotone then (2.9) and (2.10) in Proposition 3 are always satisfied with $B^+ = B^- = B$. More generally, using (2.15) we also have the following observation:

(2.20)
(a) if B^+ is quasi-monotone, (2.9) is satisfied whenever $B^+(t, \varphi) \geq B(t, \varphi)$ for $(t, \varphi) \in \mathscr{E}$, and
(b) if B^- is quasi-monotone, (2.10) is satisfied whenever $B^-(t, \varphi) \leq B(t, \varphi)$ for $(t, \varphi) \in \mathscr{E}$.

There are several implications of the techniques in Proposition 3 using quasi-monotonicity. As one such example we have

Corollary 5. *Suppose that B is quasi-monotone on $\mathscr{E}$ and* (C1)-(C6) *are valid with $S^+ = S^- = S$ and $B^+ = B^- = B$. Then for each $\chi \in \mathscr{E}$ with $v_a^- \leq \chi \leq v_b^+$ equation* (2.1) *has a (unique) solution $u(\cdot; \chi)$ on $[a, \overline{b})$, where $a < \overline{b} = \overline{b}(\chi)$. Furthermore,*

(2.21) *if $v_a^- \leq \chi \leq \psi \leq v_a^+$ then $v^-(t) \leq u(t; \chi) \leq u(t, \psi) \leq v^+(t)$ for all $t \in [a, \hat{b})$, where $\hat{b} = \min\{\overline{b}(\chi), \overline{b}(\psi)\}$.*

Proof. The fact that $u(\cdot; \chi)$ exists and satisfies $v^-(t) \leq u(t; \chi) \leq v^+(t)$ is a direct consequence of Proposition 3 by (2.20) and (2.19). Assertion (2.21) also follows from Proposition 1 by redefining $v^+(t) = u(t; \psi)$ for all $t \in [a - \tau, \overline{b}(\psi))$ and then reapplying Proposition 3.

Remark 2.5. If X is the Banach space $L^p(\overline{\Omega})^m$ or $C(\overline{\Omega})^m$, then it is natural to take

$$X_+ = \{(y_i)_1^m \in X: y_i(x) \geq 0 \text{ for } i = 1, \ldots, m \text{ and almost all } x \in \Omega\}.$$

In this case, $(z_i)_1^m \geq (y_i)_1^m$ only if $z_i(x) \geq y_i(x)$ for all $i = 1, \ldots, m$ and almost all $x \in \Omega$. It is easy to see that this partial ordering makes X into a Banach lattice.

We now indicate how these abstract results apply to Theorem 1 and Proposition 1 in the first section. In particular, we take $X = C(\overline{\Omega})^m$ and let T and S be as defined in (1.7) and (1.9), respectively. Also, B is the substitution operator defined by (1.10). Since the differentiability assertions in Theorem 1 have already been established (see the paragraph following Theorem 1), we use Corollary 4 to show the existence of a mild solution to (1.1). Therefore, set

$$K \equiv \mathscr{E}_\Lambda \equiv \{\varphi \in \mathscr{E}: \varphi(x, \theta) \in \Lambda \quad \text{for all } (x, \theta) \in \overline{\Omega} \times [-\tau, 0]\}$$

and note that (H1)–(H4) hold with $D(t) \equiv K$ since Λ (and hence K) is closed and convex. Also, property (2.4a) for S certainly holds from assumption (1.12) and the continuity property (2.3) for B is an immediate consequence of (1.5a) and (1.5b). Hence, it is sufficient to show that (1.5c) implies that B satisfies (2.4b), and so let $(t, \varphi) \in [0, \infty) \times K$. Since $h \to d(\varphi(0) + hf(t, x, \varphi); \Lambda)$ is convex on $(0, \infty)$,

$$\frac{1}{h} d(\varphi(0) + hf(t, x, \varphi); \Lambda) \downarrow 0 \quad \text{as } h \to 0+.$$

This fact along with the continuity of f shows that the limit is uniform for (t, x, φ) in each compact subset of $[0, \infty) \times \overline{\Omega} \times \mathscr{C}_\Lambda$. Since all norms on $\mathbf{R}^m$ are equivalent, we assume that $|\cdot|$ is the Euclidean norm on $\mathbf{R}^m$ and set $|y| = \max\{|y(x)| : x \in \overline{\Omega}\}$ if $y \in X \subset C(\overline{\Omega})^m$. This norm on X and the induced norm on $\mathscr{C}$ are equivalent to the original norms on X and on $\mathscr{C}$, respectively. Thus, if P_Λ is the Euclidean projection onto Λ:

$$|\xi - P_\Lambda \xi| = \min\{|\xi - \eta| : \eta \in \Lambda\},$$

then P_Λ is well defined and continuous—in fact, $|P_\Lambda \xi - P_\Lambda \eta| \le |\xi - \eta|$ for all $\xi, \eta \in \mathbf{R}^m$. Since $(x, \theta) \to \varphi(x, \theta)$ is uniformly continuous on $\overline{\Omega} \times [-\tau, 0]$ and it follows that $\{\varphi(x, \cdot) : x \in \overline{\Omega}\}$ is a compact subset of $C([-\tau, 0])^m$. hence for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if

$$y_h(x) \equiv P_\Lambda(\varphi(x, 0) + hf(t, x, \varphi(x, \cdot))) \quad \text{for } x \in \overline{\Omega},\ h > 0,$$

then $y_h \in K(\Lambda)$ and

$$\begin{aligned} d(\varphi(0) + hB(t, \varphi); K(\Lambda)) &\le |\varphi(0) + hB(t, \varphi) - y_h| \\ &\le \sup\{|\varphi(x, 0) + hf(t, x, \varphi(x, \cdot)) - y_h(x)| : x \in \overline{\Omega}\} \\ &= \sup\{d(\varphi(x, 0) + hf(t, x, \varphi(x, \cdot)); \Lambda) : x \in \overline{\Omega}\} \le h\varepsilon \end{aligned}$$

for all $0 < h \le \delta(\varepsilon)$. This shows that Theorem 1 is true.

Note that if φ and $v^\pm$ are as in (1.16) in Proposition 1, then (1.16a) may be written

$$\lim_{h \to 0+} \frac{1}{h} d(v^+(x, t) - \varphi(0) + h[f^+(t, x, v_t^+(x, \cdot)) - f(t, x, \varphi)]; [0, \infty)^m) = 0$$

and (1.16b) may be written

$$\lim_{h \to 0+} \frac{1}{h} d(\varphi(0) - v^-(x, t) + h[f(t, x, \varphi) - f^-(t, x, v_t^-(x, \cdot))]; [0, \infty)^m) = 0.$$

Comparing these statements with (2.9) and (2.10) and also comparing $(1.15)'_\pm$ in Remark 1.5 with (C4) and (C4) indicates that Proposition 1 is a direct consequence of Proposition 3. This proves all of the results in §1 on the reaction–diffusion–delay systems except those involving strict inequalities, and these are established using the results from the next section.

Remark 2.6. Here we point out that the subtangential condition (2.2) is not sufficient for invariance for functional differential equations even if D is pathwise connected. Let $X = \mathbf{R}^2$, define

$$K_1 = \{(\xi_1, \xi_2): \xi_2 \leq 0\}, \qquad K_2 = \{(\xi_1, \xi_2): \xi_1 \leq 0, \xi_2 \geq 0\}, \quad \text{and}$$
$$K_3 = \{(n^{-1}, \xi_2): \xi_2 > 0, n = 1, 2, \ldots\},$$

and set $K = K_1 \cup K_2 \cup K_3$. Then K is a closed pathwise connected subset of $\mathbf{R}^2$. For each continuous function $\nu: [-1, 0] \to R$ define

$$g(\nu) = \min\{|\nu(\theta)|: -1 \leq \theta \leq 0\}$$

and then define $f = (f_1, f_2): \mathscr{C} \to \mathbf{R}^2$ by

$$f(\varphi_1, \varphi_2) = \begin{cases} (-\nu(\varphi_2)\varphi_1(-1), 0) & \text{if } \varphi_1(-1) \leq 0, \\ (0, 0) & \text{if } \varphi_1(-1) > 0. \end{cases}$$

Taking $\chi_1(\theta) \equiv 0$ and $\chi_2(\theta) \equiv 1$ for $-1 \leq \theta \leq 0$ implies that $(\chi_1(\theta), \chi_2(\theta)) \in K$ for all $-1 \leq \theta \leq 0$, but the solution $u = (u_1, u_2)$ to

$$u' = f(u_t), \qquad t \geq 0, \ u_0 = \chi,$$

does not remain in K since $u_1'(0) > 0$ and $u_1(0) = 0$. However, if $\varphi(\theta) \in K$ for all $-1 \leq \theta \leq 0$ then

$$\lim_{h \to 0+} \frac{1}{h} d(\varphi(0) + hf(\varphi); K) = 0. \tag{2.22}$$

For if $(\varphi_1(0), \varphi_2(0)) \in K_1 \cup K_2$ then (2.32) follows easily, and if $(\varphi_1(0), \varphi_2(0)) \in K_3$ then $\varphi_1(0) = n^{-1}$ for some positive integer n. If $\varphi_1(-1) \geq 0$ then $f(\varphi) = (0, 0)$ so (3.32) is immediate, and if $\varphi_1(-1) < 0$ then there is a $\theta \in (-1, 0)$ such that $(\varphi_1(\theta), \varphi_2(\theta)) = (n^{-1}, 0)$. Hence $g(\varphi_2) = 0$ and $f(\varphi) = (0, 0)$, so (2.22) holds in this case as well.

3. Systems and strict inequalities

In this section we continue to assume that X is a Banach lattice as in the preceding section and further that m is a positive integer and there are Banach spaces X_1, $i = 1, \ldots, m$, such that $X = \prod_{i=1}^m X_i$. Suppose also that there are cones $X_i^+ \subset X_i$ such that

$$X_+ = \prod_{i=1}^m X_i^+ \text{ and if } x = (x_i)_1^m, \ y = (y_i)_1^m \in X$$
$$\text{then } x \geq y \Leftrightarrow x_i \geq y_i \text{ for all } i = 1, \ldots, m.$$

Here, of course, $\geq$ denotes the partial order on X induced by X_+ (i.e., $x \geq y \leftrightarrow x - y \in X_+$) and also the partial order on each X_i induced by X_i^+ (i.e., $x_i \geq y_i \leftrightarrow x_i - y_i \in X_i^+$). Additionally, define

$$\mathscr{C}_i = \mathscr{C}([-\tau, 0], X_i) \quad \text{and} \quad \mathscr{C}_i^+ = \{\varphi_i \in \mathscr{C}_i: \varphi_i(\theta) \in X_i^+ \text{ for } -\tau \leq \theta \leq 0\}$$

and observe that $\mathscr{C} = \prod_{i=1}^m \mathscr{C}_i$ and $\mathscr{C}_+ = \prod_{i=1}^m \mathscr{C}_i^+$ where $\mathscr{C}_+ = \{\varphi \in \mathscr{C}: \varphi(\theta) \in X_+ \text{ for } -\tau \le \theta \le 0\}$. It is assumed in this section that, in addition to (T1)–(T3) and (S1)–(S3), the following properties hold for T, S, and B:

(P1) $T(t, s)x = (T_i(t, s)x_i)_1^m$ where $T_i(t, s): X_i \to X_i$ for all $t \ge s \ge a$ and $x = (x_i)_1^m \in X$.

(P2) $S(t, s)x = (S_i(t, s)x_i)_1^m$ where $S_i(t, s): X_i \to X_i$ for all $t \ge s \ge a$ and $x = (x_i)_1^m \in X$.

(P3) $T_i(t, s): X_i^+ \to X_i^+$ and $S_i(t, s): X_i^+ \to X_i^+$ for all $t \ge s \ge a$ and $i = 1, \ldots, m$.

(P4) $B = (B_i)_1^m$ where each B_i maps $D(B)$ into X_i.

Also let $|\cdot|$ denote the norm on each X_i and assume that m_1 and M_1 are positive constants so that

$$m_1|x| \le \sum_{i=1}^m |x_i| \le M_1|x| \quad \text{for all } x = (x_i)_1^m \in X. \tag{3.1}$$

If $d(x_i, C_i) \equiv \inf\{|x_i - y_i|: y_i \in C_i\}$ for each $x_i \in X_i$ and $C_i \subset X_i$, then it is easy to see from (3.1) that

(3.2)
$$m_1 d\left(x; \prod_{i=1}^m C_i\right) \le \sum_{i=1}^m d(x_i; C_i) \le M_1 d\left(x; \prod_{i=1}^m C_i\right) \quad \text{for all } x = (x_i)_1^m \in X.$$

Observe that with the above notation equation (2.1) may be written in the component form

$$\begin{aligned} u_i(t) &= S_i(t, a)\chi_i(0) + \int_a^t T_i(t, r)B_i(r, u_r)\,dr, \qquad t > a, \\ u_i(a + \theta) &= \chi_i(\theta), \qquad -\tau \le \theta \le 0, \end{aligned} \tag{3.3}$$

where $\chi = (\chi_i)_1^m$ and $u = (u_i)_1^m$ is the solution to (2.1). Our final preliminary assumption is that the suppositions of Proposition 3 are satisfied. Hence $v^\pm = (v_i^\pm)_1^m$ are as in Proposition 3 and the solution $u = (u_i)_1^m$ to (3.3) satisfies

$$v_i^-(t) \le u_i(t) \le v_i^+(t) \quad \text{for } a \le t < \overline{b},\ i = 1, \ldots, m \tag{3.4}$$

[see (2.11) in Proposition 3]. The principal aim of this section is to develop a type of strict inequality for the solution $u = (u_i)_1^m$ and apply these results to Proposition 2 in §1.

Our first crucial assumption involves the operator $B = (B_i)_1^m$ and the operator $B^- = (B_i^-)_1^m$ [see (C5) in §2]:

(3.5) for each $R > 0$ there is an $L(R) > 0$ such that $B_i(t, \varphi) - B_i^-(t, v_t^-) \ge -L(R)[\varphi_i(0) - v_i^-(t)]$ for all $i = 1, \ldots, m$ and $(t, \varphi) \in [a, a + R] \times \mathscr{C}$ with $v_t^- \le \varphi \le v_t^+$ and $\|\varphi\| \le R$.

Note that if (3.5) holds and $(t, \varphi) \in [a, \infty) \times \mathscr{C}$ with $v_t^- \le \varphi \le v_t^+$, then

$$\varphi_i(0) - v_i^-(t) + h[B_i(t, \varphi) - B_i^-(t, v_t^-)] \ge (1 - hL(R))[\varphi_i(0) - v_i^-(t)] \ge 0$$

when $R > \max\{t-a, \|\varphi\|\}$ and $hL(R) < 1$. Thus

$$\lim_{h\to 0+} \frac{1}{h} d_i(\varphi_i(0) - v_i^-(t) + h[B_i(t, \varphi) - B^-(t, v_t^-)]; X_i^+) = 0$$

and we see immediately from (3.2) that (3.5) implies that (2.10) in Proposition 3 must hold. As is indicated by Lemma 1.1, if B is the substitution operator defined by (1.10), then (2.10) and the Lipschitz continuity of B show that (3.5) must automatically hold. The fundamental implication of (3.5) is given by the following lemma:

Lemma 3.1. *Suppose* (3.5) *holds,* $u = (u_i)_1^m$ *is the solution to* (3.3), *and* $R > 0$ *and* $a < b < \overline{b}$ *are such that* $b - a \le R$ *and* $|u(t)| \le R$ *for all* $t \in [a, b]$. *If* $L = L(R)$ *is as in* (3.5), *then*

$$u_i(t) - v_i^-(t) \ge e^{-L(t-t_0)} T_i(t, t_0)[u_i(t_0) - v_i^-(t_0)] \tag{3.6}$$

for all $a \le t_0 \le t \le b$ *and* $i = 1, \dots, m$.

Proof. First notice that using property (S2) for component S_i of S, it is easy to see that

$$u_i(t) = S_i(t, t_0)u_i(t_0) + \int_{t_0}^t T_i(t, r)B_i(r, u_r)\, dr. \tag{3.7}$$

Combining this equation with the inequality (C5) shows that if $w(t) \equiv u(t) - v^-(t)$ then

$$\begin{aligned} w_i(t) &\ge S_i(t, t_0)u_i(t_0) - S_i^-(t, t_0)v_i^-(t_0) + \int_{t_0}^t T_i(t, r)[B_i(r, u_r) - B_i^-(r, v_r^-)]\, dr \\ &\ge S_i(t, t_0)u_i(t_0) - S_i(t, t_0)v_i^-(t_0) - \int_{t_0}^t T_i(t, r)L(R)w(r)\, dr, \end{aligned}$$

where $L(R)$ is as in (3.5) [the estimate $S \ge S^-$—see (C2)—was used to obtain the last inequality]. But (S2) implies

$$S_i(t, t_0)u_i(t_0) - S_i(t, t_0)v_i^-(t_1) = T_i(t, t_0)[u_i(t_0) - v_i^-(t_0)]$$

and we see that

$$w_i(t) \ge T_i(t, t_0)w_i(t_0) - \int_{t_0}^t T_i(t, r)L(R)w(r)\, dr$$

for all $a \le t_0 \le t \le b$ and $i = 1, \dots, m$. Thus it follows that $w_i(t) \ge z_i(t)$, where

$$z_i(t) = T_i(t, t_0)w_i(t_0) - \int_{t_0}^t T_i(t, r)L(R)z_i(r)\, dr.$$

[This actually follows from Proposition 3 with $v^- \equiv z_i$, $v^+ \equiv +\infty$, $S \equiv S^- \equiv T \equiv T_i$, and $B(t, \varphi) \equiv B^-(t, \varphi) \equiv -L\varphi(0)$.] One may verify directly that

$$z_i(t) = e^{-L(t-t_0)} T_i(t, t_0)u_i(t_0)$$

is the solution and hence (3.6) is true.

In order to obtain strict inequalities for the solution to (3.3) we use positive linear functionals on X_i. So for each $i = 1, \dots, m$ let X_i^* be the dual space of X_i with $|\cdot|$ also denoting the norm on X_i^* and let

$$P_i^* = \{\Phi_i \in X_i^* : \Phi_i(x_i) \geq 0 \text{ for all } x_i \in X_i^+\}.$$

Also, let $\mathscr{R}_i$ be a nonempty indexing set and assume

$$\mathscr{R}_i^* = \{\Phi_i^\rho \in P_i^* : \rho \in \mathscr{R}_i\}$$

is a family of members of P_i^*. In addition to properties (P1)–(P4), we assume

(3.8) if $i \in \{1, \dots, m\}$ and $x_i \in X_i^+$, then $\Phi_i^\sigma(x_i) > 0$ for some $\sigma \in \mathscr{R}_i$ implies $\Phi_i^\rho(T_i(t, t_0)x_i) > 0$ for all $t > t_0 \geq a$ and all $\rho \in \mathscr{R}_i$.

Remark 3.1. If Ω is an open bounded domain in $\mathbf{R}^N$ and $X_i = C(\overline{\Omega})$ (the continuous functions from $\overline{\Omega}$ into $\mathbf{R}$ with the maximum norm), then one can take $\mathscr{R}_i = \overline{\Omega}$ and define $\Phi_i^\rho(z) = z(\rho)$ for all $z \in C(\overline{\Omega})$ and $\rho \in \overline{\Omega}$. In this case $X_+ = C_+(\overline{\Omega})$ is the set of all $z \in C(\overline{\Omega})$ such that $z(\rho) \geq 0$ for all $\rho \in \overline{\Omega}$. Notice here that

$$z \in \text{interior of } C_+(\overline{\Omega}) \leftrightarrow \Phi_i^\rho(z) > 0 \quad \text{for all } \rho \in \overline{\Omega}.$$

Furthermore, if $T_i(t, s)v_0^i$ is the solution to the heat equation with homogeneous boundary conditions as indicated by (1.7), we see that assumption (3.8) reduces to the strong maximum principle. If $X_i = C(\overline{\Omega})_0$, the space of $z \in C(\overline{\Omega})$ with $z(\rho) = 0$ for $\rho \in \partial\Omega$, then define $\mathscr{R}_i$ exactly the same with $\overline{\Omega}$ replaced by Ω and let T_i be generated by the heat equation with homogeneous Dirichlet boundary conditions. Then (3.8) is still satisfied, but the cone $C_+(\overline{\Omega})_0$ has empty interior. If $T_i(t, s)v_0^i \equiv v_0^i$ on $C(\overline{\Omega})$ or $C(\overline{\Omega})_0$ [e.g., if $i \in \Sigma_0$ in (1.7)], then we may select any $\rho \in \Omega$ and define $\mathscr{R}_i = \{\rho\}$ and $\Phi_i^0(z) = z(\rho)$ so that (3.8) is still valid. If $X_i = L^p(\Omega)$ where $1 \leq p < \infty$, then take $\mathscr{R}_i = \{(\overline{x}, r) : \overline{x} \in \Omega \text{ and } 0 < r < 1\}$ and define

$$\Phi_i^{(\overline{x}, r)}(z) = \int_{\Omega(\overline{x}, r)} z(x)\, dx \quad \text{for all } z \in L^p(\Omega)$$

where $\Omega(\overline{x}, r) = \{x \in \Omega : |x - \overline{x}| < r\}$. Again the natural cone $L_+^p(\Omega)$ of nonnegative-valued members of $L^p(\Omega)$ has empty interior, but (3.8) holds from the maximum principle when T_i is generated by the Laplacian.

Combining assumption (3.8) with Lemma 5.1 gives the following important observation:

(3.9) If (3.8) holds, $k \in \{1, \dots, m\}$, and $\Phi_k^\sigma(u_k(t_0)) > \Phi_k^\sigma(v_k^-(t_0))$ for some $t_0 \in [a, b)$ and $\sigma \in \mathscr{R}_k$, then $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $t \in (t_0, b]$ and all $\rho \in \mathscr{R}_k$.

In order to ensure that strict inequalities in one component of the solution $(u_i)_1^m$ propagate to other components, it is necessary to make further assumptions on the function $B = (B_i)_1^m$. First, assume B is quasi-positive on X_+ and consider the following property:

$$\text{(3.10)}\quad \begin{aligned} &\text{if } a \le t_1 < t_2,\ \Sigma \text{ is a nonempty, proper subset of } \{1, \dots, m\}, \text{ and} \\ &w = (w_i)_1^m : [t_1 - \tau, t_2] \to X_+ \text{ is continuous with} \\ &\quad \text{(a) } \Phi_j^\rho(w_j(t)) = 0 \text{ for all } j \in \Sigma^c,\ \rho \in \mathscr{R}_j,\ t \in [t_1 - \tau, t_2], \\ &\quad \text{(b) } \Phi_j^\rho(w_j(t)) > 0 \text{ for all } j \in \Sigma,\ \rho \in \mathscr{R}_j,\ t \in [t_1 - \tau, t_2], \\ &\text{then there are a } k \in \Sigma^c \text{ and a } \sigma \in \mathscr{R}_k \text{ such that} \\ &\qquad \sup\{\Phi_k^\sigma(B_k^k(t, w_t)) : t_1 \le t \le s\} > 0 \\ &\text{for all } t_1 < s \le t_2. \end{aligned}$$

Since $B = (B_i)_1^m$ is quasi-positive, we have that

$$\lim_{h \to 0+} \frac{1}{h} d(w_k(t) + hB_k(t, w_t); X_k^+) = 0$$

and hence

$$w_k(t) + hB_k(t, w_t) = p_h + o(h)$$

where $p_h \in X_k^+$ and $h^{-1}|o(h)| \to 0$ as $h \to 0+$. Since $k \in \Sigma^c$ we have from (3.10a) that $\Phi_k^\rho(w_k(t)) = 0$, and by the definition of P_k^* we have $\Phi_k^\rho(p_h) \ge 0$. Thus $h\Phi_k^\rho(B_k(t, w_t)) \ge \Phi_k^\rho(o(h))$ and it is immediate that if (3.10a) holds then

$$\Phi_k^\rho(B_k(t, w_t)) \ge 0 \quad \text{for all } k \in \Sigma^c,\ \rho \in \mathscr{R}_k, \text{ and } t \in [t_1, t_2].$$

Therefore the crucial point in (3.10) is that supremum is *strictly* positive for some $k \in \Sigma^c$ and $\sigma \in \mathscr{R}_k$.

Remark 3.2. Property (3.10) is connected with the concept of irreducibility and is a direct extension of property (1.18) for f in §1. In particular, if $X_i \equiv \mathbf{R}$, $\mathscr{R}_i \equiv \{1\}$, and $\Phi_i^1(\xi_i) \equiv \xi_i$ for all $i = 1, \dots, m$, then (3.10) is precisely (1.18) with $B(t, \varphi) \equiv f(t, \overline{x}, \varphi)$.

Following an approach analogous to that in §1, where (1.18) was extended to (1.22), we consider the following extension of (3.10):

(3.11)

$$\begin{aligned} &\text{if } a \le t_1 < t_2,\ \Sigma \text{ is a proper, nonempty subset of } \{1, \dots, m\}, \\ &\text{and } w = (w_i)_1^m : [t_1 - \tau, t_2] \to X \text{ is continuous with} \\ &\quad \text{(a) } v^-(t) \le w(t) \le v^+(t) \text{ for all } t_1 - \tau \le t \le t_2, \\ &\quad \text{(b) } \Phi_j^\rho(w_j(t)) = \Phi_j^\rho(v_j^-(t)) \text{ for all } j \in \Sigma^c,\ \rho \in \mathscr{R}_j, \text{ and } t_1 - \tau \le t \le t_2, \\ &\quad \text{(c) } \Phi_j^\rho(w_j(t)) > \Phi_j^\rho(v_j^-(t)) \text{ for all } j \in \Sigma,\ \rho \in \mathscr{R}_j, \text{ and } t_1 - \tau \le t \le t_2, \\ &\text{then there are a } k \in \Sigma^c \text{ and a } \sigma \in \mathscr{R}_k \text{ such that} \\ &\qquad \sup\{\Phi_k^\sigma(B_k(t, w_t) - B_k^-(t, v_t^-)) : t_1 \le t \le s\} > 0 \quad \text{for all } t_1 < s \le t_2. \end{aligned}$$

Note that (3.10) is a special case of (3.11) with $v^-(t) \equiv 0$ and $v^+(t) \equiv +\infty$. Under these conditions we have the following fundamental results regarding strict inequalities for solutions to (3.3):

Theorem 3. *Suppose that in addition to the suppositions in Proposition* 3, (P1)–(P4), (3.5), (3.8), *and* (3.11) *are satisfied. If there exists a* $t_1 \geq a$ *such that* $t_1 + (m-1)\tau < \overline{b}$ *and* $\Phi_j^\rho(u_j(t_1)) > \Phi_j^\rho(v_j^-(t_1))$ *for some* $j \in \{1, \dots, m\}$ *and some* $\rho \in \mathscr{R}_j$, *then there exists a* $t_m \in [t_1, t_1 + (m-1)\tau]$ *such that*

$$\text{(3.12)}\quad \Phi_i^\rho(u_i(t)) > \Phi_i^\rho(v_i^-(t)) \quad \textit{for all } t \in (t_m, \overline{b}), \ \textit{all } i \in \{1, \dots, m\}, \ \textit{and all } \rho \in \mathscr{R}_i.$$

Proof. For each $\overline{t} \in (a, b]$ define

$$\Gamma_+(\overline{t}) = \{i = 1, \dots, m : \Phi_i^\rho(u_i(t)) > \Phi_i^\rho(v_i^-(t)) \text{ for all } t \in [\overline{t}, b] \text{ and all } \rho \in \mathscr{R}_i\}$$

From (3.9) and continuity it follows that $i \in \Gamma_+(\overline{t})$ only in case $\Phi_i^\sigma(u_i(\overline{t})) > \Phi_i^\sigma(v_i^-(\overline{t}))$ for some $\sigma \in \mathscr{R}_i$. Suppose, for contradiction, that there are $\overline{t} \geq a + \tau$, $\varepsilon > 0$, and a nonempty, proper subset Σ of $\{1, \dots, m\}$ such that $\Gamma_+(t) = \Sigma$ for all $t \in [\overline{t} - \tau, \overline{t} + \varepsilon]$ where $\overline{t} + \varepsilon < b$. From the definition of $\Gamma_+(t)$ and (3.19) we have that

$$\text{(3.13)}\quad \begin{aligned} &\text{(a) } \Phi_i^\rho(u_i(t) - v_i^-(t)) > 0 \text{ for all } t \in [\overline{t} - \tau, \overline{t} + \varepsilon], \ i \in \Sigma, \ \rho \in \mathscr{R}_i; \\ &\text{(b) } \Phi_i^\rho(u_i(t) - v_i^-(t)) = 0 \text{ for all } t \in [\overline{t} - \tau, \overline{t} + \varepsilon], \ i \in \Sigma^c, \ \rho \in \mathscr{R}_i. \end{aligned}$$

Therefore, by assumption (3.11) there are a $k \in \Sigma^c$, a $\sigma \in \mathscr{R}_k$, and a sequence $\{\varepsilon_j\}_1^\infty$ in $(0, \varepsilon)$ such that $\varepsilon_j \to 0+$ as $j \to \infty$ and

$$\Phi_k^\sigma(B_k(\overline{t} + \varepsilon_j, u_{\overline{t}+\varepsilon_j}) - B_k^-(\overline{t} + \varepsilon_j, v_{\overline{t}+\varepsilon_j}^-)) > 0 \quad \text{for } j = 1, 2, \dots .$$

By continuity and (3.8) it may be assumed that for each j there is a number $\overline{t}_j$, $\overline{t} < \overline{t}_j \leq \overline{t} + \varepsilon_j$, such that

$$\Phi_k^\sigma(T_k(\overline{t} + \varepsilon_j, r)[B_k(r, u_r) - B_k^-(r, v_r^-)]) > 0 \quad \text{for } \overline{t}_j \leq r \leq \overline{t} + \varepsilon_j.$$

Using (C5) and equation (3.7) with $t_0 = \overline{t}_j$ and $t = \overline{t} + \varepsilon_j$, it follows that

$$\begin{aligned} \Phi_k^\sigma(u_k(\overline{t} + \varepsilon_j) - v_k(\overline{t} + \varepsilon_j)) &\geq \Phi_k^\sigma(S(\overline{t} + \varepsilon_j, \overline{t}_j)[u_k(\overline{t}_j) - v_k^-(\overline{t}_j)]) \\ &\quad + \int_{\overline{t}_j}^{\overline{t}_j + \varepsilon_j} \Phi_k^\sigma(T(\overline{t} + \varepsilon_j, r)[B_k(r, u_r) - B_k^-(r, v_r^-)])\, dr > 0. \end{aligned}$$

However, this is a contradiction to (3.13b) since $k \in \Sigma^c$. Thus we conclude that the following statement must be valid:

> if $\overline{t} \in (a, b - \tau)$ and $\Gamma_+(\overline{t})$ is nonempty, then either $\Gamma_+(\overline{t}) = \{1, \dots, m\}$ or there is a $t_0 \in (\overline{t}, \overline{t} + \tau]$ such that $\Gamma_+(\overline{t})$ is a proper subset of $\Gamma_+(t_0)$.

Using this statement, the theorem is established in the following manner: $\Gamma_+(t_1)$ is nonempty by hypothesis, so if $\Gamma_+(t_1) \neq \{1, \dots, m\}$ there is a $t_2 \in (t_1, t_1 + \tau]$ such that $\Gamma_+(t_1)$ is a proper subset of $\Gamma_+(t_2)$. If $\Gamma_+(t_2) \neq \{1, \dots, m\}$ there is a $t_3 \in (t_2, t_2 + \tau) \subset (t_1, t_1 + 2\tau]$ such that $\Gamma_+(t_2)$ is a proper subset of $\Gamma_+(t_3)$.

Continuing in this manner, it is easy to see that there is a $t_m \le t_1 + (m-1)\tau$ such that $\Gamma_+(t_m) = \{1, \dots, m\}$, and hence the theorem is proven.

These ideas also have immediate implications for strict inequalities between comparable solutions to (3.3). In place of (3.5) we assume

for each $R > 0$ there is an $L(R) > 0$ such that

$$B_i(t, \varphi) - B_i(t, \psi) \ge -L(R)[\varphi_i(0) - \psi_i(0)]$$

for all $i = 1, \dots, m$ and $(t, \varphi), (t, \psi) \in [a, a+R] \times \mathscr{C}$ with $v_t^- \le \psi \le \varphi \le v_t^+$ and $\|\psi\|, \|\varphi\| \le R$.

This condition implies that B is quasi-monotone; for if $v_t^- \le \psi \le \varphi \le v_t^+$ and $hL(R) < 1$ then

$$\varphi_i(0) - \psi_i(0) + h[B_i(t, \varphi) - B_i(t, \psi)] \ge (1 - hL(R))[\varphi_i(0) - \psi_i(0)] \ge 0$$

and it is immediate that

$$\lim_{h \to 0+} \frac{1}{h} d_i(\varphi_i(0) - \psi_i(0) + h[B_i(t, \varphi) - B_i(t, \psi)]; X_i^+) = 0.$$

This combined with (3.2) shows that (2.19) must hold, and hence B must be quasi-monotone whenever (3.14) is satisfied. The condition on B corresponding to (3.11) is the following:

(3.15)
if $a \le t_1 < t_2 \le b$, Σ is a nonempty, proper subset of $\{1, \dots, m\}$, and $w^\pm = (w_i^\pm)_1^m : [t_1 - \tau, t_2] \to X$ are continuous with

(a) $v^+(t) \ge w^+(t) \ge w^-(t) \ge v^-(t)$ for all $t \in [t_1 - \tau, t_2]$,

(b) $\Phi_i^\rho(w_i^+(t)) = \Phi_i^\rho(w_i^-(t))$ for all $i \in \Sigma^c$, $\rho \in \mathscr{R}_i$, $t \in [t_1 - \tau, t_2]$,

(c) $\Phi_i^\rho(w_i^+(t)) > \Phi_i^\rho(w_i^-(t))$ for all $i \in \Sigma$, $\rho \in \mathscr{R}_i$, $t \in [t_1 - \tau, t_2]$,

then there are a $k \in \Sigma^c$ and a $\sigma \in \mathscr{R}_k$ such that

$$\sup\{\Phi_k^\sigma(B_k(t, w_t^+)) - \Phi_k^\sigma(B_k(t, w_t^-)) : t_1 \le t \le s\} > 0$$

for all $t_1 < s \le t_2$.

Corollary 6. *In addition to the suppositions in Theorem* 3, *suppose that* (3.14) *and* (3.15) *are satisfied. Also, assume* $\chi^\pm \in \mathscr{C}$ *with* $v_a^- \le \chi^- \le \chi^+ \le v_a^+$ *and let* $u^\pm$ *be the solution to* (3.3) *on* $[a, \overline{b}(\chi^\pm))$ *with* $\chi = \chi^\pm$, *respectively. Then*

$$v^-(t) \le u^-(t) \le u^+(t) \le v^+(t) \quad \text{for all } a \le t \le \hat{b} \tag{3.16}$$

where $\hat{b} = \min\{\overline{b}(\chi^-), \overline{b}(\chi^+)\}$. *Also, if there is a* $t_1 \ge a$ *such that* $t_1 + (m-1)\tau < \hat{b}$ *and* $\Phi_j^\sigma(u_j^+(t_1)) > \Phi_j^\sigma(u_j^-(t_1))$ *for some* $j \in \{1, \dots, m\}$ *and some* $\sigma \in \mathscr{R}_j$, *then there is a* $t_m \in [t_1, t_1 + (m-1)\tau]$ *such that*

$$\Phi_j^\rho(u_j^+(t)) > \Phi_j^\rho(u_j^-(t)) \quad \text{for all } t \in (t_m, \hat{b}),\ \text{all } i \in \{1, \dots, m\}, \text{ and all } \rho \in \mathscr{R}_j. \tag{3.17}$$

Proof. Since these suppositions imply that those of Corollary 5 hold, we have that (3.16) is valid directly from (2.21). Assertion (3.17) now follows directly from Theorem 3 with $v^- \equiv u^-$ and $u \equiv u^+$.

Under certain circumstances the differential of B may be used to determine if (3.14) and (3.15) are valid. So assume that $B(t, \cdot)$ is continuously Fréchet differentiable and let $dB(t, \varphi) = (dB_i(t, \varphi))_1^m$ denote the F-derivatives of $B(t, \cdot)$ at φ: $dB(t, \varphi)$ is a bounded linear map from $\mathscr{C}$ into X such that

$$\lim_{\psi \mapsto 0} \frac{|B(t, \varphi + \psi) - B(t, \varphi) - dB(t, \varphi)\psi|}{\|\psi\|} = 0.$$

We have the following criteria:

Lemma 3.2. *Suppose that $B(t, \cdot)$ is continuously Fréchet differentiable for each $a < t < \infty$ and that the F-derivative $dB(t, \varphi)$ of $B(t, \cdot)$ at φ satisfies the following*:

(3.18) *for each $R > 0$ there is an $L(R) > 0$ such that $dB_i(t, \varphi)\psi \geq -L(R)\psi_i(0)$ for all $\psi \in \mathscr{C}$, $i \in \{1, \dots, m\}$ and $(t, \varphi) \in [a, a+R] \times \mathscr{C}$ with $v_t^- \leq \varphi \leq v_t^+$ and $\|\varphi\| \leq R$.*

(3.19) *Property (3.10) is valid with $B(t, \cdot)$ replaced by $dB(t, \varphi)$ for all but an at most countable number of $(t, \varphi) \in [a, \infty) \times \mathscr{C}$ with $v_t^- \leq \varphi \leq v_t^+$.*

Then B satisfies properties (3.14) and (3.15).

Proof. Since $B(t_0, \cdot)$ is C^1 and the set of φ in $\mathscr{C}$ such that $v_t^- \leq \varphi \leq v_t^+$ is convex, we have

$$B_i(t_0, \varphi) - B_i(t_0, \psi) = \int_0^1 dB_i(t_0, r\varphi + (1-r)\psi)(\varphi - \psi)\, dr \tag{3.20}$$

for all $i = 1, \dots, m$ and $v_t^- \leq \psi \leq \varphi \leq v_t^+$. Since

$$dB_i(t_0, r\varphi + (1-r)\psi)(\varphi - \psi) \geq -L(R)(\varphi_i(0) - \psi_i(0))$$

if $R = \max\{t_0 - a, \|\varphi - \psi\|\}$ by (3.18), it is immediate that B must satisfy property (3.14). Now assume that Σ, t_1, t_2, and $w^{\pm}$ are as in (3.15). Applying the statements between (3.10) and Remark 3.2 with B_j replaced by $dB_j(t, rw_t^+ + (1-r)w_t^-)$ and $w = w^+ - w^-$ shows that

$$\Phi_j^\rho(dB_j(t, rw_t^+ + (1-r)w_t^-)(w_t^+ - w_t^-)) \geq 0 \tag{3.21}$$

for all $j \in \Sigma^c$, $0 \leq r \leq 1$, $t \in [t_1, t_2]$, and $\rho \in \mathscr{R}_j$. If $t_1 < s \leq t_2$ then (3.19) [using (3.10) with B_k replaced by $dB_k(s, rw_s^+ + (1-r)w_s^-)$] implies there is an $r_0 \in (0, 1)$ such that

$$\Phi_k^\sigma(dB_k(s, r_0 w_s^+ + (1-r_0)w_s^-)(w_s^+ - w_s^-)) > 0 \tag{3.22}$$

for some $k \in \Sigma^c$ and $\sigma \in \mathscr{R}_k$. By continuity (3.22) holds for all r_0 in some open subinterval of $(0, 1)$. Setting $i = k$, $t_0 = s$, $\varphi = w_s^+$, and $\psi = w_s^-$ in (3.20) shows that

$$\Phi_k^\sigma(B_k(s, w_s^+) - B_k(s, w_s^-)) = \int_0^1 \Phi_k^\sigma(dB_k(s, rw_s^+ + (1-r)w_s^-)(w_s^+ - w_s^-))\,dr.$$

But (3.22) and (3.21) imply that this integral must be strictly positive, and hence B must also satisfy (3.15). This completes the proof.

The final topic of this section is to show how these techniques imply the results on strict inequalities stated in §1. In particular, it suffices to give a proof of Proposition 2. So assume the hypotheses in Proposition 2 hold, let $X_i = C(\overline{\Omega})$ for $i = 1, \dots, m$, and let $T = (T_i)_1^m$, $S = (S_i)_1^m$, and $B = (B_i)_1^m$ be defined as in §1 [see (1.7), (1.9), and (1.10)]. Clearly (P1)–(P4) hold and (1.25) in Lemma 1.1 implies that (3.5) is also satisfied. Set

$$X_i^+ \equiv C(\overline{\Omega})_+ \equiv \{y_i \in C(\overline{\Omega}) : y_i(x) \geq 0 \quad \text{for all } x \in \overline{\Omega}\}$$

and define the indexing sets $\mathscr{R}_i$ by $\mathscr{R}_i = \{\overline{x}\}$ if $i \in \Sigma_0^c$ and $\mathscr{R}_i = \overline{\Omega}$ if $i \in \Sigma_0^c$. Then for each $i \in \{1, \dots, m\}$ define $\Phi_i^\rho(y_i) \equiv y_i(\rho)$ for all $\rho \in \mathscr{R}_i$ and note that (3.8) holds in this case (use the strong maximum principle for $i \in \Sigma_0^c$). Also, let $w = (w_i)_1^m : \overline{\Omega} \times [t_1 - \tau, t_2] \to \mathbf{R}$ be continuous [so that $t \to w(\cdot, t)$ is continuous from $[t_1 - \tau, t_2]$ into $C(\overline{\Omega})^m$] and let (a)–(c) in (3.11) hold. Taking $z_j(t) \equiv w_j(\overline{x}, t)$ for all $t \in [t_1 - \tau, t_2]$ and $j \in \{1, \dots, m\}$, we see immediately from (1.22) that there is a $k \in \Sigma^c$ with

$$\sup\{f_k(t, \overline{x}, w(\overline{x}, \cdot)) - f_k^-(t, \overline{x}, v^-(\overline{x}, \cdot)) : t_1 \leq t \leq s\} > 0$$

for each $t_1 < s \leq t_2$. But this says precisely that

$$\sup\{\Phi_k^\sigma(B_k(t, w_t) - B_k^-(t, v_t^-)) : t_1 \leq t \leq s\} > 0$$

for all $t_1 < s \leq t_2$ when $\sigma = \overline{x}$, and hence (3.11) is also satisfied. Therefore, each of the suppositions in Theorem 3 is fulfilled and we see that (1.23) is an immediate consequence of (3.12) in this case. This proves Proposition 2.

4. Existence proofs

The purpose of this section is to give detailed proofs. The basic existence result is stated in Theorem 2 of §2, and it is assumed throughout that (T1)–(T3), (S1)–(S3), (H1)–(H4), and (2.2) are satisfied. In place of (T3), it is often more convenient to use a Lyapunov-like function in our estimates, and its basic properties are described in our first lemma:

Lemma 4.1. *For each* $(t, x) \in [a, \infty) \times X$ *define*

$$W[t, x] = \sup[\{e^{-\omega r}|T(t+r, t)x| : r \geq 0\}.$$

Then $W[t, \cdot]$ *is a norm on* X *satisfying*

(4.1)
(a) $|x| \leq W[t, x] \leq \hat{M}|x|$ *for* $(t, x) \in [a, \infty) \times X$ *and*
(b) $W[t, T(t, s)x] \leq e^{\omega(t-s)} W[s, x]$ *for* $x \in X$ *and* $t \geq s \geq a$.

In particular, if $\hat{M}=1$ *then* $W[t,x]=|x|$ *for all* $t\geq a$, $x\in X$.

Proof. It is easy to see that $W[t,\cdot]$ is a norm and that (4.1a) holds [where $\hat{M}$ and ω are as in (T3)]. Furthermore, if $t\geq s\geq a$ then

$$\begin{aligned} W[t,T(t,s)x] &= \sup\{e^{-\omega r}|T(t+r,s)x| : r\geq 0\} \\ &= e^{\omega(t-s)}\sup\{e^{-\omega(t+r-s)}|T(s+(t+r-s),s)x| : r\geq 0\} \\ &\leq e^{\omega(t-s)}\sup\{e^{-\omega\rho}|T(s+\rho,s)x| : \rho\geq 0\} \\ &= e^{\omega(t-s)}W[s,x] \end{aligned}$$

and we see that (4.2b) also is true. This completes the proof of the lemma.

In order to establish existence we construct approximate solutions and then show that these approximate solutions converge to a solution. So let $(a,\chi)\in\mathscr{D}$ be given and consider the initial value problem

$$\text{(4.2)}\qquad \begin{aligned} &u(t)=S(t,a)\chi(0)+\int_a^t T(t,r)B(r,u_r)\,dr, \qquad a\leq t\leq\sigma, \\ &u(a+\theta)=\chi(\theta) \quad \text{for } -\tau\leq\theta\leq 0. \end{aligned}$$

Now let $\hat{K}=\hat{K}(\sigma+\varepsilon_0)$ and $\eta=\eta_{\sigma+\varepsilon_0}$ be as in (H3), $\hat{M}$, ε be as in (T3), and select the numbers $M\geq\hat{M}$, R, N, $\varepsilon_0>0$, and $\sigma>a$ such that

(4.3)

(a) $\varepsilon_0 M<R/3$ and $\eta(\varepsilon_0)<R/3$;

(b) $|B(t,\varphi)|\leq N$ whenever $(t,\varphi)\in\mathscr{D}$ with $t\in[a,\sigma+\varepsilon_0]$, $\varphi(\theta)=\chi(t-a+\theta)$ if $-\tau\leq\theta\leq 0$ and $t+\theta\leq a$, and $|\varphi(\theta)-\chi(0)|\leq(2K+1)R$ if $-\tau\leq\theta\leq 0$ and $t+\theta>a$;

(c) $|S(t,a)\chi(0)-\chi(0)|\leq R/3$ and $\|T(t,s)\|\leq M$ if $a\leq s\leq t\leq\sigma+\varepsilon_0$;

(d) $e^{|\omega|(\sigma+\varepsilon_0-a)}\hat{M}(MN+\varepsilon_0)(\sigma+\varepsilon_0-a)<R/3$.

Because of the continuity of B and S there exist numbers so that (4.3) is satisfied. We show that for each $\varepsilon\in(0,\varepsilon_0]$ we may construct ε-approximate solutions to (4.2) on $[a,\sigma]$. This construction is established with several lemmas and it is assumed that (4.3) holds in each of these lemmas.

Lemma 4.2. *Suppose that* $\{t_i\}_0^n$ *is an increasing sequence in* $[a,\sigma+\varepsilon_0]$, $\{x_i\}_0^n$ *is a sequence in* X, *and there is a number* $\overline{M}>0$ *such that*

$$|S(t_{i+1},t_i)x_i-x_{i+1}|\leq\overline{M}(t_{i+1}-t_i) \quad \textit{for } i=0,1,\ldots,n-1.$$

Then

$$|S(t_i,t_0)x_0-x_i|\leq e^{|\omega|(t_i-t_0)}\hat{M}\overline{M}(t_i-t_0)$$

for $i=0,1,\ldots,n$.

Proof. If W is as in Lemma 4.1, it suffices from (4.1a) to show that

$$W[t_i,S(t_i,t_0)x_0-x_i]\leq e^{|\omega|(t_i-t_0)}\hat{M}\overline{M}(t_i-t_0) \tag{4.4}$$

for $i = 0, \ldots, n$. This is established by induction: Clearly (4.4) holds if $i = 0$, so assume that (4.4) holds for some $i \in \{0, \ldots, n-1\}$. Then, using the fact that $W[t_{i+1}, \cdot]$ is a norm, (S2), and (4.1), we have

$$\begin{aligned}
&W[t_{i+1}, S(t_{i+1}, t_0)x_0 - x_{i+1}] \\
&\quad \le W[t_{i+1}, S(t_{i+1}, t_i)S(t_i, t_0)x_0 - S(t_{i+1}, t_i)x_i] \\
&\qquad + W[t_{i+1}, S(t_{i+1}, t_i)x_i - x_{i+1}] \\
&\quad \le W[t_{i+1}, T(t_{i+1}, t_i)(S(t_i, t_0)x_0 - x_i)] + \hat{M}|S(t_{i+1}, t_i)x_i - x_{i+1}| \\
&\quad \le e^{\omega(t_{i+1}-t_i)} W[t_i, S(t_i, t_0)x_0 - x_i] + \hat{M}\overline{M}(t_{i+1} - t_i) \\
&\quad \le e^{|\omega|(t_{i+1}-t_i)}\{e^{|\omega|(t_i-t_0)}\hat{M}\overline{M}(t_i - t_0)\} + \hat{M}\overline{M}(t_{i+1} - t_i) \\
&\quad \le e^{|\omega|(t_{i+1}-t_0)}\hat{M}\overline{M}(t_{i+1} - t_0).
\end{aligned}$$

This establishes (4.4) and completes the proof.

Now let ε be in $(0, \varepsilon_0]$. The ε-approximate solution w and a corresponding increasing sequence $\{t_i\}_0^\infty$ are constructed in the following manner: Set $t_0 = a$ and define $w(a+s) = \chi(s)$ for $-\tau \le s \le 0$. Assume that i is a nonnegative integer and w is constructed and continuous on $[a-\tau, t_i]$ where $a \le t_i < \sigma+\varepsilon_0$ and $w(t) \in D(t)$ on $[a-\tau, t_i]$. If $t_i \ge \sigma$ set $t_{i+1} = t_i$ and if $t_i < \sigma$ choose $\delta_i \in [0, \varepsilon]$ as follows:

(4.5)

(a) $|S(t, t_i)w(t_i) - w(t_i)| \le \varepsilon$ if $t_i \le t \le t_i + \delta_i$;

(b) $d(S(t_i+\delta_i, t_i)w(t_i) + \int_{t_i}^{t_i+\delta_i} T(t_i+\delta_i, r)B(t_i, w_{t_i})\,dr;\ D(t_i+\delta_i)) \le \varepsilon\delta_i/2$;

(c) $|w(t) - w(s)| \le \varepsilon$ if $t, s \in [a-\tau, t_i]$ with $|t-s| \le \delta_i$;

(d) if (a), (b), and (c) hold with δ_i replaced by $\eta > 0$, then $\delta_i > \eta/2$.

The continuity of w and S along with the subtangential condition (2.2) implies that $\delta_i > 0$ if $t_i < \sigma$. Define $t_{i+1} = t_i+\delta_i$ and by (4.5b) select $w(t_{i+1}) \in D(t_{i+1})$ so that

$$\left| S(t_{i+1}, t_i)w(t_i) + \int_{t_i}^{t_{i+1}} T(t_{i+1}, r)B(t_i, w_{t_i})\,dr - w(t_{i+1}) \right| \le \varepsilon(t_{i+1} - t_i). \tag{4.6}$$

Also, by assumption (H3) define w on (t_i, t_{i+1}) so that w is continuous, $w(t) \in D(t)$, and

$$|w(t) - w(s)| \le \eta(|t-s|) + \hat{K}\frac{|w(t_{i+1}) - w(t_i)|}{t_{i+1} - t_i}|t-s| \quad \text{if } t_i \le s,\ t \le t_{i+1}. \tag{4.7}$$

Lemma 4.3. *With the above notations, the following are true*:

(a) $|w(t_i) - \chi(0)| \le 2R/3$ *for* $i = 0, 1, 2, \ldots$;

(b) $|w(t) - \chi(0)| \le (2\hat{K}+1)R$ *for* $t \in [a, t_i]$, $i = 0, 1, 2, \ldots$;

(c) $|B(t_i, w_{t_i})| \le N$ *for* $i = 0, 1, 2, \ldots$.

Proof. We prove (a), (b), and (c) simultaneously by induction on i. Clearly all three hold for $i = 0$ [see (5.3b)], so assume $k \geq 0$ is given and that (a), (b), and (c) are true whenever $0 \leq i \leq k$. By (4.6), (c), and (4.3c),

$$(4.8) \quad \begin{aligned} |S(t_{i+1}, t_i)w(t_i) - w(t_{i+1})| &\leq \left| \int_{t_i}^{t_{i+1}} T(t_{i+1}, r)B(t_i, w_{t_i})\, dr \right| + \varepsilon(t_{i+1} - t_i) \\ &\leq (MN + \varepsilon)(t_{i+1} - t_i) \end{aligned}$$

for all $i = 0, \ldots, k$. Therefore, by Lemma 4.2 with $n = k+1$, $\overline{M} = (MN+\varepsilon)$, and $x_i = w(t_i)$, we see that

$$\begin{aligned} |S(t_{k+1}, a)\chi(0) - w(t_{k+1})| &\leq e^{|\omega|(t_{k+1}-a)} \hat{M}(MN + \varepsilon)(t_{k+1} - a) \\ &< R/3 \end{aligned}$$

by (4.3d). Consequently, by (4.3c) we have

$$\begin{aligned} |w(t_{k+1}) - \chi(0)| &\leq |w(t_{k+1}) - S(t_{k+1}, a)\chi(0)| + |S(t_{k+1}, a)\chi(0) - \chi(0)| \\ &\leq R/3 + R/3 = 2R/3 \end{aligned}$$

so (a) holds with $i = k + 1$. Also, if $t_k \leq t \leq t_{k+1}$ when using (4.7), (4.3a), and part (a) of this lemma, we see that

$$\begin{aligned} |w(t) - \chi(0)| &\leq |w(t) - w(t_k)| + |w(t_k) - \chi(0)| \\ &\leq \eta(t - t_k) + \hat{K}\frac{|w(t_{k+1}) - w(t_k)|}{t_{k+1} - t_k}(t - t_k) + 2R/3 \\ &\leq \eta(t_{k+1} - t_k) + \hat{K}|w(t_{k+1}) - w(t_k)| + 2R/3 \\ &\leq \eta(\varepsilon) + \hat{K}\{|w(t_{k+1}) - \chi(0)| + |\chi(0) - w(t_k)|\} + 2R/3 \\ &\leq R/3 + 2\hat{K}(2R/3) + 2R/3 \leq (2\hat{K} + 1)R. \end{aligned}$$

This shows that (b) is true for $i = k + 1$ and since (b) now implies

$$|B(t_{k+1}, w_{t_{k+1}})| \leq N$$

by (4.3b), we have that this lemma is true by induction.

Since the sequence $\{t_i\}_0^\infty$ is nondecreasing in $[a, \sigma + \varepsilon_0]$ we have that $\rho = \lim_{i\to\infty} t_i$ exists and w is defined on $[a, \rho)$ with $w(t) \in D(t)$ for all $t \in [a, \rho)$. We also have the following:

Lemma 4.4. *If $\rho \in (a, \sigma + \varepsilon_0]$ and $w: [a - \tau, \rho) \to X$ is as above, then $z = \lim_{t\to\rho-} w(\rho)$ exists and $z \in D(\rho)$. In particular, if $w(\rho) \equiv z$ then $w_t \to w_\rho$ in $\mathscr{C}$ as $t \to \rho-$.*

Proof. First we show that $w(t_i)$ converges as $i \to \infty$ (note that this lemma is obvious if $t_n \geq \sigma$ for some n, since the construction of $\{t_i\}_0^\infty$ implies that $t_i = t_n$ for all $i \geq n$). Using (4.8) in the proof of Lemma 4.3, we have that

$$|S(t_{i+1}, t_i)w(t_i) - w(t_{i+1})| \leq \overline{M}(t_{i+1} - t_i) \quad \text{for } i = 0, 1, 2, \ldots$$

where $\overline{M} = (MN + \varepsilon)$. Therefore, if $j > k \geq 0$, we have using Lemma 4.2 with $n = j - k$, $x_i = w(t_{k+1})$, and t_i replaced by t_{k+i} that

$$|S(t_j, t_k)w(t_k) - w(t_j)| \leq e^{|\omega|(t_j - t_k)}\hat{M}\overline{M}(t_j - t_k) \leq \overline{N}(\rho - t_k)$$

where $\overline{N} = e^{|a|(p-a)}\hat{M}\overline{M}$. Therefore, let $\bar{\varepsilon} > 0$ be given and choose $k > 0$ so that $2\overline{N}(\rho - t_k) < \bar{\varepsilon}/2$. Since $S(t_i, t_k)w(t_k) \to S(\rho, t_k)w(t_k)$ as $i \to \infty$ by the continuity of S, choose $n(\bar{\varepsilon}) > k$ so that

$$|S(t_i, t_k)w(t_k) - S(t_j, t_k)w(t_k)| \leq \bar{\varepsilon}/2 \quad \text{if } i, j \geq n(\bar{\varepsilon}).$$

Then for $i, j \geq n(\bar{\varepsilon})$ we have

$$\begin{aligned} |w(t_i) - w(t_j)| &\leq |w(t_i) - S(t_i, t_k)w(t_k)| \\ &\quad + |S(t_i, t_k)w(t_k) - S(t_j, t_k)w(t_k)| + |S(t_j, t_k)w(t_k) - w(t_j)| \\ &\leq \overline{N}(\rho - t_k) + \bar{\varepsilon}/2 + \overline{N}(\rho - t_k) < \bar{\varepsilon}. \end{aligned}$$

Thus $\{w(t_i)\}_0^\infty$ is Cauchy and $z = \lim_{i\to\infty} w(t_i)$ exists. Since $(t_i, w(t_i)) \in D$ and D is closed, it follows that $(\rho, z) \in D$ and hence $z \in D(\rho)$. Since (4.7) shows that

$$\begin{aligned} |w(t) - w(t_i)| &\leq \eta(|t - t_i|) + \hat{K}\frac{|w(t_{i+1}) - w(t_i)|}{t_{i+1} - t_i}|t - t_i| \\ &\leq \eta(t_{i+1} - t_i) + \hat{K}|w(t_{i+1}) - w(t_i)| \end{aligned}$$

it is immediate that $w(t) \to z$ as $t \to \rho-$ and the assertions in this lemma now follow.

The next lemma shows that the above construction always results in the existence of an ε-approximate solution on $[a - \tau, \sigma]$ where σ is independent of ε.

Lemma 4.5. *If $\{t_i\}_0^\infty$ is as above, then there is an integer $n = n(\varepsilon)$ such that $t_n \geq \sigma$.*

Proof. Suppose, for contradiction, that no such n exists. Then $t_i < \sigma$ for all $i \geq 1$ and since $w(t_i) \to z$ as $i \to \infty$ by the preceding lemma, it follows from the continuity of w and S that (4.5a) and (4.5c) hold with δ_i independent of i. Hence, there are a $\delta > 0$ and a $k \geq 1$ such that (4.5a) and (4.5c) hold with δ_i replaced by $\rho + \eta - t_i$ for $i \geq k$ and $0 < \eta \leq \delta$. If $\eta \in (0, \delta]$ and $i = i(\eta) \geq k$ is sufficiently large so that $2\delta_i < \rho + \eta - t_i$, then it follows from (4.5d) and (4.5b) that

$$\begin{aligned} d\left(S(\rho + \eta, t_i)w(t_i) + \int_{t_i}^{\rho+\eta} T(p + \eta, r)B(t_i, w_{t_i})\,dr; D(\rho + \eta)\right) \\ > (\rho + \eta - t_i)\varepsilon/2. \end{aligned}$$

Using continuity and letting $i \to \infty$ it follows that

$$d\left(S(\rho + \eta, \rho)z + \int_{\rho}^{\rho+\eta} T(\rho + \eta, r)B(\rho, w_\rho)\,dr; D(\rho + \eta)\right) \geq \eta\varepsilon/2$$

for each $\eta \in (0, \delta]$. This is, of course, a contradiction to the subtangential condition (2.2) and shows that this lemma is true.

Now let $\{\varepsilon_n\}_1^\infty$ be a decreasing sequence in $(0, \varepsilon_0)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and for each $n \geq 1$ let w^n and $\{t_i^n\}_{i=1}^\infty$ be as constructed above with $\varepsilon = \varepsilon_n$, $t_i = t_i^n$, and $w = w^n$. We know from Lemma 4.5 that for each $n \geq 1$ there is an $n_0 = n_0(n)$ such that $t_{n_0}^n \geq \sigma$. For convenience we define a companion function v^n for w^n in the following manner:

(4.9) $v^n(t) = S(t, a)\chi(0) + \int_a^t T(t, r)B(\gamma^n(r), w^n_{\gamma^n(r)})\,dr$ for $t \in [a, \sigma]$ and $v^n(a + \theta) = \chi(\theta)$ for $-\tau \leq \theta \leq 0$, where $\gamma^n : [a, \sigma] \to [a, \sigma]$ satisfies $\gamma^n(t) = t_i^n$ whenever $t_i^n \leq t < t_{i+1}^n$.

Observe that if $a \leq s \leq t \leq \sigma$ we have from (S2) that

$$\begin{aligned} v^n(t) &= S(t, s)S(s, a)\chi(0) + T(t, s)\int_a^s T(s, r)B(\gamma^n(r), w^n_{\gamma^n(r)})\,dr \\ &\quad + \int_s^t T(t, r)B(\gamma^n(r), w^n_{\gamma^n(r)})\,dr \\ &= S(t, s)\left[S(s, a)\chi(0) + \int_a^s T(s, r)B(\gamma^n(r), w^n_{\gamma^n(r)})\,dr\right] \\ &\quad + \int_s^t T(t, r)B(\gamma^n(r), w^n_{\gamma^n(r)})\,dr \end{aligned}$$

and it follows that

$$v^n(t) = S(t, s)v^n(s) + \int_s^t T(t, r)B(\gamma^n(r), w^n_{\gamma^n(r)})\,dr \quad \text{for } a \leq s \leq t \leq \sigma. \tag{4.10}$$

There is the following important estimate for w^n and v^n:

Lemma 4.6. *Suppose that ε_n, w^n, and v^n are as above for $n = 1, 2, \ldots$. Then there exists a $\hat{P} > 0$, independent of n, such that*

$$|v^n(t) - w^n(t)| \leq \hat{P}\max\{\varepsilon_n, \eta(\varepsilon_n)\} \quad \textit{for } a - \tau \leq t \leq \sigma \textit{ and } n = 1, 2, \ldots. \tag{4.11}$$

Proof. Let W be as in Lemma 4.1 and let $n \geq 1$. We first show by induction on i that if $a \leq t_i^n \leq \sigma$ then

$$W[t_i^n, v^n(t_i^n) - w^n(t_i^n)] \leq e^{|w|(t_i^n - a)}\hat{M}\varepsilon_n(t_i^n - a). \tag{4.12}$$

For convenience we omit the superscript n and set

$$\Phi(t, s) = \int_s^t T(t, r)B(\gamma(r), w_{\gamma(r)})\,dr \quad \text{for } a \leq s \leq t \leq \sigma.$$

Clearly (4.12) holds for $i = 0$, so assume that it is valid for some $i \geq 0$ where

$t_i < t_{i+1} \le \sigma$. Then by (4.10), (4.1), and the fact that $W[t_{i+1}, \cdot]$ is a norm,

$$\begin{aligned} W[t_{i+1}, v(t_{i+1}) - w(t_{i+1})] &= W[t_{i+1}, S(t_{i+1}, t_1)v(t_i) + \Phi(t_{i+1}, t_i) - w(t_{i+1})] \\ &\le W[t_{i+1}, S(t_{i+1}, t_i)v(t_i) - S(t_{i+1}, t_i)w(t_i)] \\ &\quad + W[t_{i+1}, S(t_{i+1}, t_i)w(t_i) + \Phi(t_{i+1}, t_i) - w(t_{i+1})] \\ &\le W[t_{i+1}, T(t_{i+1}, t_i)(v(t_i) - w(t_i))] \\ &\quad + \hat{M}|S(t_{i+1}, t_i)w(t_i) + \Phi(t_{i+1}, t_i) - w(t_{i+1})| \\ &\le e^{|w|(t_{i+1}-t_i)} W[t_i, v(t_i) - w(t_i)] + \hat{M}\varepsilon_n(t_{i+1} - t_i) \end{aligned}$$

where (4.6) was used in the last estimate. By the induction hypothesis

$$\begin{aligned} W(t_{i+1}, v(t_{i+1}) - w(t_{i+1})] \le e^{|\omega|(t_{i+1}-t_i)} e^{|\omega|(t_i-a)} \hat{M}\varepsilon_n(t_i - a) \\ + \hat{M}\varepsilon_n(t_{i+1} - t_i) \end{aligned}$$

and it is immediate that (4.12) holds with i replaced by $(i+1)$. Thus (4.12) is valid by induction on i, so by (4.1a) we have that

$$|v(t_i) - w(t_i)| \le e^{|\omega|(\sigma-a)} \hat{M}(\sigma - a)\varepsilon_n \equiv \hat{Q}\varepsilon_n \tag{4.13}$$

where we continue to omit the superscript n on v, w, and t_i. If $t_i \le t < t_{i+1} \le \sigma$ then applying (4.10), (4.5a), (4.7), (4.13), and (c) in Lemma 4.3

$$\begin{aligned} |v(t) - w(t)| &\le |S(t, t_i)v(t_i) + \Phi(t, t_i) - S(t, t_i)w(t_i)| \\ &\quad + |S(t, t_i)w(t_i) - w(t_i)| + |w(t_i) - w(t)| \\ &\le M|v(t_i) - w(t_i)| + |\Phi(t, t_i)| + \varepsilon_n \\ &\quad + \hat{K}(t - t_i) + \hat{K}|w(t_{i+1}) - w(t_i)| \\ &\le M\hat{Q}\varepsilon_n + MN\varepsilon_n + \varepsilon_n + \eta(\varepsilon_n) + \hat{K}|w(t_{i+1}) - w(t_i)|. \end{aligned}$$

But (4.8) and (4.5a) imply

$$\begin{aligned} |w(t_{i+1}) - w(t_i)| &\le |w(t_{i+1}) - S(t_{i+1}, t_i)w(t_i)| + |S(t_{i+1}, t_i)w(t_i) - w(t_i)| \\ &\le (MN + \varepsilon_n)(t_{i+1} - t_i) + \varepsilon_n \le (MN + \varepsilon_n + 1)\varepsilon_n \end{aligned}$$

and we see that (4.11) is valid by combining the two preceding inequalities

Lemma 4.7. *There exists a constant $\hat{Q} > 0$, independent of $t \in [a, \sigma]$ and $n \ge 1$, such that*

$$\|w_t^n - w_{\gamma^n(t)}^n\| \le \hat{Q}\max\{\varepsilon_n, \eta(\varepsilon_n)\} \quad \textit{for } t \in [a, \sigma] \textit{ and } n = 1, 2, \ldots,$$

where $\|\cdot\|$ is the norm in $\mathscr{C}$.

Proof. Suppose that $n \ge 1$ and $t_i^n \le t \le t_{i+1}^n \le \sigma$. Then

$$\|w_t^n - w_{\gamma^n(t)}^n\| = \|w_t^n - w_{t_i^n}^n\| = \sup\{|w^n(t+s) - w^n(t_i^n + s)| : -\tau \le s \le 0\}.$$

Since $|(t+s) - (t_i^n + s)| = |t - t_i^n| \le t_{i+1}^n - t_i^n$, we have from (4.5c) that

$$|w^n(t+s) - w^n(t_i^n + s)| \le \varepsilon_n \quad \text{whenever } t + s \le t_i^n.$$

If $t+s > t_i^n$ then $-s < t - t_i^n \le t_{i+1}^n - t_i^n$; so $|w^n(t_i^n) - w^n(t_i^n + s)| \le \varepsilon_n$ by (4.5c). Therefore, by (4.7) and (4.14),

$$\begin{aligned}|w^n(t+s) - w^n(t_i^n+s)| &\le |w^n(t+s) - w^n(t_i^n)| + |w^n(t_i^n) - w^n(t_i^n+s)| \\ &\le \eta(\varepsilon_n) + \hat{K}|w^n(t_{i+1}^n) - w^n(t_i^n)| + \varepsilon_n \\ &\le \eta(\varepsilon_n) + \hat{K}(MN + \varepsilon_n + 1)\varepsilon_n + \varepsilon_n\end{aligned}$$

whenever $t+s > t_i^n$. Combining these estimates establishes the lemma.

Lemma 4.8. *Suppose that there is a function* $u: [a-\tau, \sigma] \to X$ *such that*

$$u(t) = \lim_{n\to\infty} w^n(t) \quad \textit{uniformly for } t \in [a-\tau, \sigma].$$

Then $(t, u(t)) \in D$ *and* u *is a solution to* (4.2) *on* $[a, \sigma]$.

Proof. Since D is closed and $(t, w^n(t)) \in D$ we have that $(t, u(t)) \in D$ for all $t \in [a-\tau, \sigma]$. By Lemma 4.6 $v^n(t) \to u(t)$ uniformly on $[a-\tau, \sigma]$ as $n \to \infty$ and by Lemma 4.7,

$$\begin{aligned}\|w_{\gamma^n(t)}^n - u_t\| &\le \|w_{\gamma^n(t)}^n - w_t^n\| + \|w_t^n - u_t\| \\ &\le \hat{Q}\max\{\varepsilon_n, \eta(\varepsilon_n)\} + \sup\{|w^n(t+s) - u(t+s)|: -\tau \le s \le 0\}\end{aligned}$$

and it follows that $w_{\gamma^n(t)}^n \to u_t$ in $\mathscr{C}$ as $n \to \infty$, uniformly for $t \in [a, \sigma]$. By the continuity of B we have

$$B(\gamma^n(t), w_{\gamma^n(t)}^n) \to B(t, u_t) \quad \text{as } n \to \infty$$

and this limit is also uniform for $t \in [a, \sigma]$. Thus, by (4.9), (4.11), and the continuity of T,

$$\begin{aligned}u(t) &= \lim_{n\to\infty}\left\{S(t, a)\chi(0) + \int_a^t T(t, r)B(\gamma^n(r), w_{\gamma^n(r)}^n)\,dr\right\} \\ &= S(t, a)\chi(0) + \int_a^t T(t, r)B(r, u_r)\,dr\end{aligned}$$

and u is a solution to (4.2).

Proof of existence in Theorem 2. Since $|t - \gamma^n(t)| \le \varepsilon_n$ it follows that $|\gamma^n(t) - \gamma^m(t)| \to 0$ as $n, m \to \infty$, uniformly for $t \in [a, \sigma]$. Therefore, by (1.4), Lemma 4.7, and Lemma 4.6 it follows that $\overline{R}$ is such that $|w^n(t)| \le \overline{R}$ for all $n \ge 1$ and $t \in [a-\tau, \sigma]$, then

$$\begin{aligned}&|B(\gamma^n(r), w_{\gamma^n(r)}^n) - B(\gamma^m(r), w_{\gamma^m(r)}^m)| \\ &\quad \le L_{\overline{R}}\|w_{\gamma^n(r)}^n - w_{\gamma^m(r)}^m\| + \nu(\gamma^n(r) - \gamma^m(r)) \\ &\quad \le L_{\overline{R}}\|w_r^n - w_r^m\| + \overline{\varepsilon}_{n,m} \le L_{\overline{R}}\|v_r^n - v_r^m\| + \hat{\varepsilon}_{n,m}\end{aligned}$$

where $\overline{\varepsilon}_{n,m}, \hat{\varepsilon}_{n,m} \to 0$ as $n, m \to \infty$. Using (4.9) and (T3), we have

$$|v^n(t) - v^m(t)| \le \int_a^t ML_{\overline{R}}\|v_r^n - v_r^m\|\,dr + \eta_{n,m}$$

where $\eta_{n,m} \to 0$ as $n, m \to \infty$. Defining

$$q_{n,m}(t) = \max\{|v^n(r) - v^m(r)| : a - \tau \le r \le t\}$$

we see that for each $t \in [a, \sigma]$ there is an $r(t) \in [a - \tau, t]$ such that

$$\begin{aligned} q_{n,m}(t) &= |v^n(r(t)) - v^m(r(t))| \\ &\le \int_a^{r(t)} ML_R \|v_r^n - v_r^m\| \, dr + \eta_{n,m} \\ &\le \int_a^t ML_R q_{n,m}(r) \, dr + \eta_{n,m}. \end{aligned}$$

Gronwall's inequality along with the fact that $\eta_{n,m} \to 0$ as $n, m \to \infty$ shows that $q_{n,m}(t) \to 0$ as $n, m \to \infty$, and hence $\{v^n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a - \tau, \sigma]$. This implies that $\{w^n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy on $[a - \tau, \sigma]$ and hence (4.2) has a solution by Lemma 4.8. This establishes the existence of a solution under the suppositions of Theorem 2, and because of the Lipschitz continuity of B the uniqueness assertion in Theorem 2 follows using standard techniques and we omit the proof.

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